Baku Mathematical Journal
2025, Vol. 4, No 1, PP. 77-94
https://doi.org/10.32010/j.bmj.2025.07

A NONLINEAR CROSS DIFFUSION MODEL ANALYSIS BY USING HOLLING TYPE III FUNCTIONAL RESPONSE TO DESCRIBE FISH AND ZOOPLANKTON DYNAMICS TAKING INTO ACCOUNT THE FISHING EFFECT IN THE ECOSYSTEM

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Received: 07.01.2025 / Revised: 19.02.2025 / Accepted: 26.02.2025

Abstract. *In this paper, we analyse a mathematical model of a cross-diffusion and self-diffusion system describing the dynamics of fish and zooplankton population. A type III Holling functional response was used in the formulation of the model taking into account the effect of fishing in the ecosystem. Satisfactory results concerning the local and global existence properties of the solutions. Persistence conditions as well as the local stability of equilibrium states are established. We have shown in particular that, under certain conditions, cross-diffusion can trigger the emergence of spatial patterns. Numerical experiments are performed to visualize complex spatial patterns and dynamic behaviour in different fish exploited areas.*

Keywords: prey predator model, fishing effort, Holling type III functional response, persistence, global solution existence and stability, zooplankton and fish ecosystem, fishing exploited areas

Mathematics Subject Classification (2020): 42B20, 42B25, 42B35

1. Introduction

Mathematical modelling involves using mathematics to describe, explain or predict behaviour or phenomena in the real world. This modelling can be particularly useful for

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exploring questions or testing ideas in complex systems. A mathematical model is an abstraction of a physical system using a precise language to describe the behaviour of the system. Modelling is the conversion of a concrete problem from the real world into terms of a mathematical nature, [1], [4]-[6]. Ecology describes the relationships between organisms, and between them and their environment, thus determining aspects such as the abundance of species, the compositions of a biological community which represents a group of organisms in space and in time. Structured models have been increasingly used in biology for several decades in various fields such as population dynamics, epidemiology, evolutionary ecology, but also to describe populations, [1], [2], [6], [14], [15] Modelling in biology does not date from today, Fibonacci proposed a model in 1202 to describe the growth of a population of rabbits, but the real birth of this discipline is more around the 15th century with, to name but a few, Euler, Daniel Bernoulli, [4], [6], [8], [15]. The reaction-diffusion systems taking into account cross-diffusion phenomena are relevant in the modelling of numerous phenomena, in various disciplines and in particular in population dynamics. Many mathematical problems concerning these systems are still open, particularly in the field of fisheries resource management. The theory of spatial pattern generation dates back to the pioneering work of Turing in 1952. These authors demonstrate that when the diffusion of the inhibitor is greater than that of the activator, the concentration can evolve from the initial quasi-homogeneity towards a formation of inhomogeneous pattern, [6]-[9], [14]. This situation implies that the equilibrium of the nonlinear system is asymptotically stable in the absence of diffusion but unstable in the presence of diffusion. By adding diffusion to the halieutic system, theoretically demonstrated that diffusion plays an important role in generating spatial patterns by Wendkouni Ouedraogo and al [14], [15].

In the field bio-diversity study in particular case of resource halieutic, several mathematical models have been developed to describe the dynamics of marine system, [1], [2], [10]. Many models assume a logistic equation for fish stock with a catch term proportional to stock and fishing effort, [1], [2], [10], [15]. The authors have published numerous works which aim to describe the dynamics of fishing models in which several time scales are involved. Despite the mathematical simplifications provided by these differences in time scales, the dynamics of fisheries can be very complex [1], [10], [15]. Recently, there has been an increasing interest in studying spatial predator- prey models [10], [14] to describe fish and plankton population taking into account cross and self diffusion. In such cases, spatio-temporel dynamics of an ecological system is represented by a coupled nonlinear reaction-diffusion equations.

In this paper, we study cross and self diffusion fish zooplankton model with Holling type III functional responses taking into account fishing effect in our study.

This manuscript is organized as follows. In the section 2, we present the mathematical model formulation which will be the subject of our study. In the section 3, we prove the local and global solution of the models. Moreover, we establish the local stability of the non-negative constant equilibrium, by using the linearisation method and analysing the corresponding characteristic equations. Numerical experiments are performed in the section 4 and finally, in the last section, we end with some conclusion remarks and future works.

2. Mathematical Model Formulation

The fish-zooplankton spatio-temporal dynamics of an ecological system is represented by a system of coupled non-linear reaction-diffusion equations theory, [13], [14].

2. 1. Original model formulation

In the general case, the dynamics of fish (T^f) and zooplankton (T^p) is governed by the ordinaries differential equation model at any time $t > 0$ is written as follows, [8], [10], [14]:

$$\begin{cases} \dfrac{dT^p}{dt} &= \pi_1(T^p) - j_2(T^p, F)T^p, \\ \dfrac{dT^f}{dt} &= j_3(T^p, T^f)T^f - j_4(T^p, T^f)T^f, \end{cases} \tag{1}$$

- $\pi_1, j_2, j_3, j_4 \in C^\infty$ and are positive,
- $\pi_1(T^p)$, the growth function of the zooplankton population,
- $j_2(T^p, T^f)$ is the amount of prey consumed by a predator per time unit,
- $j_3(T^p, T^f)$ represents the rate of conversion of the prey into predator,
- $j_4(T^p, T^f)$ is the predator mortality rate due to the fishing.

If we fix the functions values intervening in the model (1), we can illustrate the resulting dynamics of zooplankton-fish model, [13], [14] with the diagram in Fig.1.

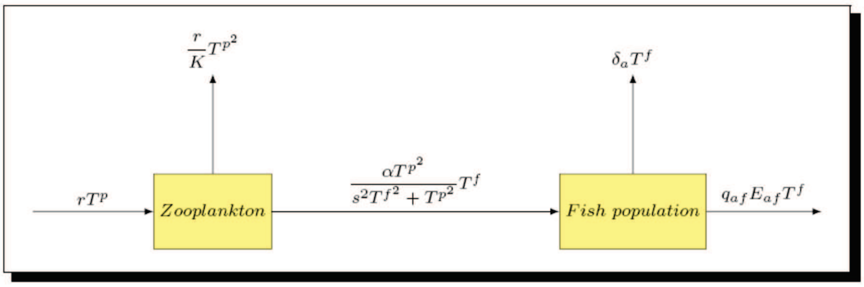

Fig. 1. Compartmental ecological representation for zooplankton-fish model taking into account the fishing effect in the ecosystem with nonlinear functional response

Namely, according to the above assumptions and the interaction diagram in figure 1, the final ODE system used to model the evolution of prey and predators over time is given by:

$$\begin{cases} \dfrac{dT^p}{dt} = rT^p\left(1 - \dfrac{T^p}{K}\right) - \dfrac{\beta T^{p^2}}{s^2T^{f^2} + T^{p^2}}T^f, \quad T^p(0) = T_0^p > 0, \\ \dfrac{dT^f}{dt} = -\delta_a T^f + \dfrac{\alpha T^{p^2}}{s^2T^{f^2} + T^{p^2}}T^f - q_{af}E_{af}T^f, \quad T^f(0) = T_0^f > 0. \end{cases} \tag{2}$$

- r : zooplankton growth rate,
- β : mortality rate due to competition between the individuals of the zooplankton,
- α : capture rate of the fish population on the zooplankton,
- K : carrying capacity for the zooplankton,
- δ_a : fish mortality rate,
- E_{af} : fishing effort on the fish population,
- q_{af} : capturability coefficient of the fishing,
- s : fish half-saturation constant for a Holling type III functional response.

2. 2. Self diffusion model formulation

We introduce the self diffusion effect in the model (2). Let us consider $\Omega \subset \mathbb{R}^n \quad (n \geq 1)$ and the following diffusion terms for $x \in \Omega$:

- $\delta_{a1}(x)$ zooplankton population self-diffusion terms,
- $\delta_{a2}(x)$ fish population self-diffusion terms.

By considering the work established in [13], [14], the reaction diffusion model associated with the model (1) can be modelled for $x \in \Omega$, $t > 0$ as follows:

$$\begin{cases} \partial_t T^p - \operatorname{div}(\delta_{a1}(x)\nabla T^p) = \pi_1(T^p) - j_2(T^p, F)T^p, \\ \\ \partial_t T^f - \operatorname{div}(\delta_{a2}(x)\nabla T^f) = j_3(T^p, T^f)T^f - j_4(T^p, T^f)T^f. \end{cases} \tag{3}$$

The zero-flux boundary condition is considered in our modelling.

$$\delta_{ai}(x)\nabla B(x,t).\nu(x) = 0, \quad i = 1, 2, \quad x \in \partial\Omega, \quad t > 0, \quad B = T^p, \ T^f,$$

where ν is the unit vector normal to $\partial\Omega$ and the initial positive and bounded conditions

$$B(x, 0) = B_0(x) > 0, \quad B = T^p, T^f, \quad x \in \Omega.$$

We make the following assumptions:

(H_{ac}) : all parameters of the system (3) are positive constants.
(H_{bc}) : diffusion coefficients of the system (3) are independent of x.

By taking $\delta_{a1}(x) = \delta_{a1}, \ \ \delta_{a2}(x) = \delta_{a2}$, according to (H_{ac}) and if the hypothesis (H_{bc}) is satisfy, the model obtained previously becomes:

$$\begin{cases} \partial_t T^p &= \ rT^p\left(1 - \dfrac{T^p}{K}\right) - \dfrac{\beta {T^p}^2}{s^2 {T^f}^2 + {T^p}^2} T^f + \delta_{a1}\Delta T^p, \qquad x \in \Omega, t > 0, \\ \\ \partial_t T^f &= \ -\delta_a T^f + \dfrac{\alpha {T^p}^2}{s^2 {T^f}^2 + {T^p}^2} T^f - q_{af}E_{af}T^f + \delta_{a2}\Delta T^f, \qquad x \in \Omega, t > 0. \end{cases} \tag{4}$$

δ_{a1} and δ_{a2} represent the natural dispersive force of movement of individuals. The homogeneous Neumann boundary condition means that model (4) is self-contained and has no population flux across the boundary.

2. 3. Self and cross diffusion model formulation

We introduce the cross diffusion effect in the model (4). More precisely, the movement of a fish population at any particular location is influenced by the gradient of the concentration of the zooplankton population at that location, and the movement of the prey is affected by the gradient of the concentration of the fish population at the same location. Such scenario can be mathematically described with cross diffusion [3], [6], [13], [14], we have for any $(x, t) \in \Omega\times]0, +\infty[$:

$$\begin{cases} \partial_t T^p = rT^p \left(1 - \dfrac{T^p}{K}\right) - \dfrac{\beta {T^p}^2}{s^2 {T^f}^2 + {T^p}^2} T^f + \delta_{a1} \Delta T^p, \\ \\ \partial_t T^f = -\delta_a T^f + \dfrac{\alpha {T^p}^2}{s^2 {T^f}^2 + {T^p}^2} T^f - q_f E_f T^f + \delta_{a2} \mathrm{div}((1 + \delta_{a3} T^p) \nabla T^f + \delta_{a3} T^f \nabla T^p). \end{cases} \tag{5}$$

The diffusion terms $\delta_{a2}\delta_{a3}$ are referred as cross-diffusion pressure, describing a mutual interference between individuals population. The system (5) means that, in addition to the dispersive force, the diffusion of fish population also depends on population pressure from zooplankton population. We can rewrite as follows: $\delta_{a2}\mathrm{div}((1 + \delta_{a3}T^p)\nabla T^f + \delta_{a3}T^f \nabla T^p) = \delta_{a2}\Delta[(1 + \delta_{a3}T^p)T^f]$ and regard $\delta_{a2}(1 + \delta_{a3}T^p)\nabla T^f + \delta_{a2}\delta_{a3}T^p\nabla T^p$ as the flux of the fish population. If $\delta_{a3} > 0$, the term $\delta_{a2}\delta_{a3}T^p\nabla T^p$ of the flux of the predator is directed toward the decreasing population density of zooplankton. So the final cross and self diffusion model can write for any $(x, t) \in \Omega\times]0, +\infty[$ as follows:

$$\begin{cases} \partial_t T^p = rT^p \left(1 - \dfrac{T^p}{K}\right) - \dfrac{\beta {T^p}^2}{s^2 {T^f}^2 + {T^p}^2} T^f + \delta_{a1} \Delta T^p, \\ \\ \partial_t T^f = -\delta_a T^f + \dfrac{\alpha {T^p}^2}{s^2 {T^f}^2 + {T^p}^2} T^f - q_{af} E_{af} T^f + \delta_{a2} \Delta[(1 + \delta_{a3} T^p) T^f]. \end{cases} \tag{6}$$

3. Mathematical Results

We choose the scaling by non-dimensionlizing for to reduce the number of parameters, [2], [13], [14]: $T^p \to KU_a$, $T^f \to \frac{K}{s}V_a$, $t \to \frac{t}{r}$, $x \to x\left(\dfrac{r}{K\delta_{a1}}\right)^{\frac{1}{2}}$, with the scaling, system (2) and (6) takes respectively the following form:

$$\begin{cases} \dfrac{dU_a}{dt} = U_a\left(1 - U_a\right) - \dfrac{e_1 V_a U_a^2}{U_a^2 + V_a^2} := H_1(U_a, V_a), \quad U_a(0) = U_{a_0} > 0, \\ \\ \dfrac{dV_a}{dt} = -e_2 V_a + \dfrac{e_3 V_a U_a^2}{U_a^2 + V_a^2} := H_2(U_a, V_a), \quad V_a(0) = V_{a_0} > 0, \end{cases} \tag{7}$$

$$\begin{cases} \partial_t U_a - D_{a1}\varDelta U_a = U_a\left(1-U_a\right) - \dfrac{e_1 V_a U_a^2}{U_a^2+V_a^2}, \; x\in\varOmega, \quad t>0, \\ \\ \partial_t V_a - D_{a2}\varDelta[(1+D_{a3}U_a)V_a] = -e_2 V_a + \dfrac{e_3 V_a U_a^2}{U_a^2+V_a^2}, \; x\in\varOmega, \quad t>0, \end{cases} \tag{8}$$

$$e_1 = \frac{\beta}{sr}, \; e_2 = \frac{\delta_a + q_{af}E_{af}}{r}, \; e_3 = \frac{\alpha}{r}, \; D_{a1} = \frac{\delta_{a1}}{r}, \; D_{a2} = \frac{\delta_{a2}}{r} \text{ and } D_{a3} = \delta_{a3}K.$$

3. 1. Partial results for the ODE system

Proposition 1. *The fish-zooplankton nonlinear system* (7) *admits an only maximal solution* $(U_a(t), V_a(t))$ *defined on an interval* $[0, T_{max}[$. *Therefore,* $\{U_a \geq 0, V_a \geq 0\}$ *is positively invariant for the system* (7).

Proof. In fact, the theorem of Cauchy-Lipschistz, [2], [5], [12] ensures the existence and uniqueness of local solution of the system (7) and since we have $H_1(0, V_a) = 0$, for $V_a \geq 0$ and $H_2(U_a, 0) = 0$ for $U_a \geq 0$, then the set $\{U_a \geq 0, V_a \geq 0\}$ is positively invariant for the system (7). ◀

We consider the following hypothesis:

(H_{a1}) : $e_2 > max\{e_2, C_a\}$ with $C_a = e_1\sqrt{e_2e_3 - e_2^2}$ and (H_{a2}) : $\dfrac{2e_2C_a - e_2^2}{2e_2e_3(e_3-e_2)} < 1.$

Proposition 2. *[13], [14]. The following result gives the stationeries states and their existence condition if* (H_1) *is satisfied.*

(i) *The stationery state* $E_{a0} = (0,0)$ *is the non-negative constant equilibrium. This equilibrium is still unstable.*

(ii) *The stationery state* $E_{a1} = (1,0)$ *is a non-negative constant equilibrium point corresponding to extinction of the fish population. This equilibrium is a saddle point with locally stable manifold in the* U_a*-direction and with locally unstable manifold in the* V_a*-direction when* $e_3 > e_2$*, or is locally asymptotically stable in the* $U_a - V_a$*-plane when* $e_3 < e_2$.

(iii) *The stationery state* $E_{a2} = (U_a^*, V_a^*)$ *a non-negative constant equilibrium point of the system, corresponding to the coexistence of zooplankton and fish population with:* $U_a^* = 1 - C/e_3$ *and* $V_a^* = U_a^*\sqrt{(e_2-e_2)/e_2}$.

Theorem 1. *Assume that the hypothesis* (H_{a1}) *and* (H_{a2}) *are satisfied, then the interior equilibrium point* (U_a^*, V_a^*) *of the system* (7) *is locally asymptotically stable.*

Proof. To study the asymptotic stability of the non trivial point E_{a2}, we are needed to define

$$\mathbf{\Sigma}_{\mathbf{U}_a} = \begin{pmatrix} U_a\left(1-U_a\right) - \dfrac{e_1V_aU_a^2}{U_a^2+V_a^2} \\ \\ -e_2V_a + \dfrac{e_3V_aU_a^2}{U_a^2+V_a^2} \end{pmatrix},$$

$$\mathbf{\Sigma}_{\mathbf{U_a}}(U_a^*) = \begin{pmatrix} \dfrac{-e_3^2 + 2e_2C_a}{e_3^2} & \dfrac{e_1e_2(e_3 - 2e_2)}{e_3^2} \\ \dfrac{2e_2(e_3 - e_2)C_a}{e_1e_3} & -\dfrac{2e_2(e_3 - e_2)}{e_3} \end{pmatrix}.$$

Let us consider $\mathrm{tr}(\mathbf{\Sigma}_{\mathbf{U_a}}(U_a^*))$ and $\det(\mathbf{\Sigma}_{\mathbf{U_a}}(U_a^*)$ defined by

$$\mathrm{tr}(\mathbf{\Sigma}_{\mathbf{U_a}}(U_a^*)) = \frac{2e_2C_a - e_2^2 - 2e_2e_3(e_3 - e_2)}{e_3^2} \text{ and } \det(\mathbf{\Sigma}_{\mathbf{U_a}}(U_a^*) = \frac{2e_2(e_3 - e_2)(e_3 - C_a)}{e_3^2}.$$

The characteristic equation of $\mathbf{\Sigma}_{\mathbf{U_a}}(U_a^*)$ is define as follows: $\mu^2 - tr(\mathbf{\Sigma}_{\mathbf{U_a}}(U^*)\mu + \det(\mathbf{\Sigma}_{\mathbf{U_a}}(U_a^*) = 0$. It is easy to verify that $\mathrm{tr}(\mathbf{\Sigma}_{\mathbf{U_a}}(U_a^*) < 0$ and $\det(\mathbf{\Sigma}_{\mathbf{U_a}}(U_a^*) > 0$ under the hypothesis (H_{a1}) and (H_{a2}). So, by using the Routh-Hurwitz stability criterion, E_{a2} is locally asymptotically stable for the system (7), [2], [6], [10]-[12]. ◄

3. 2. System fish-zooplankton without cross diffusion

We consider system (8) without cross-diffusion in the following form:

$$\begin{cases} \partial_t U_a - D_{a1}\Delta U_a = U_a(1 - U_a) - \dfrac{e_1 V_a U_a^2}{U_a^2 + V_a^2}, & x \in \Omega, \quad t > 0, \\ \partial_t V_a - D_{a2}\Delta V_a = -e_2 V_a + \dfrac{e_3 V_a U_a^2}{U_a^2 + V_a^2}, & x \in \Omega, \quad t > 0, \\ \partial_\nu U_a = \partial_\nu V_a = 0, \ x \in \partial\Omega, \ t > 0, \ U_a(x,0) = U_{a0} > 0, \ V_a(x,0) = V_{a0} > 0, \ x \in \Omega. \end{cases} \tag{9}$$

We are established the global existence and uniqueness of classical solutions of system (9) by proving that for all time $T, \|U_a(.,t)\|_{\mathbf{L}^\infty}$ and $\|V_a(.,t)\|_{\mathbf{L}^\infty}$ are bounded for $0 \leq t < T$. Let us consider $U_a(x,t) = (U_{a1}(x,t), ..., U_{am}(x,t)), V_a(x,t) = (V_{a1}(x,t), ..., V_{am}(x,t)), \quad m \geq 1$ and $\rho = (\rho_1, ..., \rho_m) > 0$ (inequalities between vectors are satisfied coordinate-wise), vector functions $p(x,t;U_a) = (p_1(x,t;U_a), ..., p_m(x,t;U_a))$ is continuously differentiable and quasi-monotonically with respect to $\partial_{U_{aj}} p_i(x,t;U_a) \geq 0, i, j = 1, .., m, i \neq j$. We use a bounded domain with smooth boundary $\Omega \subset \mathbb{R}^n$ and we assume the following condition:

(Ha) : We assume hear that $U_a, V_a \in \mathcal{C}^2$ in $x \in \Omega$, and a class $\mathcal{C}^1$ in $(x,t) \in \Omega \times [0,\infty)$.
(Hb) : $U_{at} - \rho\Delta U_a - p(x,t;U_a) \leq V_{at} - \rho\Delta V_a - p(x,t;V_a)$, where $(x,t) \in \Omega \times [0,\infty)$.
(Hc) : The quantity $\partial_\nu U_a(x,t) = \partial_\nu V_a(x,t) = 0$, for any $(x,t) \in \partial\Omega \times [0,\infty)$.

Firstly we are introduced the following comparison theorem and a basic lemma.

Theorem 2. *[2], [6], [10], [11], [13]. Assume that the hypothesis* $(Ha) - (Hb) - (Hc)$ *hold,then we have* $U_a(x,t) \leq V_a(x,t)$ *for any* $(x,t) \in \overline{\Omega} \times [0,\infty)$.

Lemma 1. *[2], [6], [13]. Let us consider the semi-linear parabolic problem*

$$\begin{cases} \beta_1 \partial_t Q_a - \beta_2 \Delta Q_a = Q(\beta_3 - \beta_4 Q_a), \ x \in \Omega, \ t > 0, \\ \\ Q_a(x,0) = Q_{a0}(x) \geq 0, \ x \in \Omega, \ \dfrac{\partial Q_a(x,t)}{\partial \nu} = 0, \ x \in \partial\Omega, \ t > 0, \end{cases} \tag{10}$$

where we have β_i is positive constant for $i = 1, 2, 3, 4$. It is known that there is unique positive solution Q_a of (10)*, satisfying $\lim_{t \to \infty} Q_a(x,t) = \beta_3/\beta_4$ uniformly in $x \in \Omega$.*

Theorem 3. *[2], [11], [13]. Let us consider (U_{a0}, V_{a0}) a nonegative initial data in $\mathbf{L}^\infty(\Omega)$, such that,*

$$U_{a0}(x) \leq 1 \quad \textit{and} \quad V_{a0}(x) \leq e_3/e_2.$$

Her the domain Ω is a bounded in $\mathbb{R}^n, n \leq 3$, with a boundary $\partial\Omega$ of class $\mathbf{C}^{2+\epsilon}$, for $\epsilon > 0$. Then, there exists a unique non-negative classical solution of the model (9) *for all $(x,t) \in \Omega \times [0, \infty)$.*

Proof. [2], [7], [10], [11], [13]. We prove the local existence of solutions using the well-known semigroup theory. Note that H_1 and H_2 are continuously differentiable and from the assumptions on the initial condition, the system (9) has a unique classical solution (U_a, V_a), for $(x,t) \in \Omega \times [0, T_{max})$. Moreover, if $T_{max} < \infty$, then

$$\lim_{t \to T_{max}} \sup_{x \in \Omega} \{|U_a(x,t)| + |V_a(x,t)|\} = \infty.$$

For to prove the positivity of solutions of the system (9), observe that the functions H_1 and H_2 satisfy

$$H_1(0, V_a) \geq 0 \quad \text{and} \quad H_2(U_a, 0) \geq 0, \quad \text{for} \quad \text{all} \quad U \geq 0, \quad V \geq 0,$$

and by assumption the initial data $(U_{a0}(x), V_{a0}(x))$ are in $[0, \infty) \times [0, \infty)$ for any $x \in \Omega$. By using the maximum principle [6], [12], [16], the solution $U_a(x,t), V_a(x,t))$ is in $[0, \infty) \times [0, \infty)$ for all $x \in \Omega$ and for all $t > 0$. The set $[0, \infty) \times [0, \infty)$ is positively invariant for the system (9).

Let as consider $W_{a1}(t) = \dfrac{W_{a01}}{W_{a01} + (1 - W_{a01})e^{-t}}$ the solution of the initial value problem, [2], [13] define by

$$\begin{cases} \dfrac{\partial W_{a1}}{\partial t} = W_{a1}(1 - W_{a1}), \ t > 0, \\ W_{a1}(0) = W_{a01} \leq 1. \end{cases}$$

By using the following inequality problem

$$\begin{cases} \dfrac{\partial U_a}{\partial t} - \Delta U_a - U_a(1 - U_a) \leq 0 = \dfrac{\partial W_{a1}}{\partial t} - \Delta W_{a1} - W_{a1}(1 - W_{a1}), \ x \in \Omega, \ t > 0, \\ \\ \dfrac{\partial U_a}{\partial \nu} = \dfrac{\partial W_{a1}}{\partial \nu} = 0, \ x \in \partial\Omega, \ t > 0, \\ \\ U_a(0,x) = U_{a0}(x) \leq W_{a01} \ \equiv \ \max_{\overline{\Omega}} U_{a0}(x). \end{cases}$$

Using the Theorem 2, we conclude that $U_a(x,t) \leq W_{a1}(x,t) \leq 1$. Therefore $U_a(x,t) \leq 1$ for all $0 \leq t \leq T$ and $x \in \Omega$. In a similar way, as $V_{a0}(x) \leq e_3/e_2$ and $V_a(x,t) \leq 1$ for all $0 \leq t \leq T$ and $x \in \Omega$, we conclude that the function $V_a(x,t) \leq e_3/e_2$ for all $0 \leq t \leq T$. Finally, we also conclude that the domain $[0,1) \times [0, e_3/e_2)$ is a positively invariant region for global solutions of system (9). ◀

We continuous our study concerning to the dissipation and the uniform persistence of (9). We first introduce some important remark.

Remark 1. [2], [11], [12], [13]. We introduce the two following important remark for the rest of this paper.

1. System (9) said to be dissipative if for any nonnegative initial data $(U_{a0}(x), V_{a0}(x))$ with $U_{a0}(x) \neq 0, V_{a0}(x) \neq 0$, there exists a positive constant $\kappa = \kappa(U_0, V_0)$ such that the solution $(U_a(x,t), V_a(x,t))$ of (9) satisfies
$$\limsup_{t\to\infty} max_{\overline{\Omega}} U_a(.,t) \leq \kappa, \quad \limsup_{t\to\infty} max_{\overline{\Omega}} V_a(.,t) \leq \kappa.$$
2. System (9) said to be uniformly persistent if for any nonnegative initial data $(U_{a0}(x), V_{a0}(x))$ with $U_{a0}(x) \neq 0, V_{a0}(x) \neq 0$, there exists a positive constant $\rho = \rho(U_{a0}, V_{a0})$ such that the solution $(U_a(x,t), V_a(x,t))$ of (9) satisfies
$$\liminf_{t\to\infty} min_{\overline{\Omega}} U_a(.,t) \geq \rho, \quad \liminf_{t\to\infty} min_{\overline{\Omega}} V_a(.,t) \geq \rho.$$

Theorem 4. *[2], [6], [10], [13]. We have the following results:*

(a) If (U_a, V_a) is the solution of (9), then we have:
$$\limsup_{t\to\infty} max_{\overline{\Omega}} U_a(.,t) \leq 1 \quad and \limsup_{t\to\infty} max_{\overline{\Omega}} V_a(.,t) \leq e_3/e_2. \tag{11}$$

(b) The system (9) is uniformly persistent if $e_1 < 1$.

Proof. [12], [16].

We assume that (U_a, V_a) is any solution of (9) with $U_{a0}(x) \geq 0, V_{a0}(x) \geq 0$ and $U_{a0}(x) \neq 0, V_{a0}(x) \neq 0$. We have also $U_a(.,t) > 0$ for all $t \geq 0$ and $V_a(.,t) > 0$ for all $t \geq 0$. It is important to note that the first element U_a satisfies:
$$\begin{cases} \dfrac{\partial U_a}{\partial t} - \Delta U_a \leq U_a(1-U_a), \ x \in \Omega, \ t > 0, \\ \\ \dfrac{\partial U_a}{\partial \nu} = 0, x \in \partial\Omega, \ t > 0, \ U_a(0,x) = U_{a0}(x) \geq 0, \ x \in \Omega. \end{cases}$$
The initial value problem of ordinary differential equation as follows :
$$\begin{cases} \dfrac{\partial z_a}{\partial t} = z_a(1-z_a), \ t > 0, \\ \\ z_a(0) = max_{\overline{\Omega}} U_{a0}(.) > 0. \end{cases}$$
The first inequality of (11) is followed by the standard comparison principle since $\lim_{t\to\infty} z_a(t) \leq 1$. So, for any $\varepsilon > 0$, there exists $T > 0$, such that $U_a(x,t) \leq 1 + \varepsilon$ for all $x \in \Omega$ and $t \geq T$. We proceed as follows of V_a. ◀

In order to study the locally asymptotic stability of the equations (9), we set up the following notations. In fact we need introduce some notations, [2], [6], [11], [12], [16].

Let us consider $0 = \lambda_0 < \lambda_1 < \lambda_2 < ...$ the eigenvalues of the operator $-\Delta$ in Ω. We use the homogeneous Neumann boundary condition and $R(\lambda_j)$ be the space of the eigenfunctions corresponding to λ_j in $\mathbf{C^1}(\overline{\Omega})$ for $j = 0, 1, 2, ...$; $\varphi_{jk}; k = 1, ..., \dim[R(\lambda_j)]$ be an orthonormal basis of $R(\lambda_j)$, and $\mathbf{X_{ajk}} = \{\mathbf{c}\varphi_{jk} : \mathbf{c} \in \mathbb{R}^2\}$. Let us define the set

$$\mathbf{X_a} = \left\{ (U_a, V_a)^T \in [\mathbf{C^1}(\overline{\Omega})]^2 : \frac{\partial U_a}{\partial \nu} = \frac{\partial V_a}{\partial \nu} = 0, \text{ on } \partial\Omega \right\}$$

and consider the decomposition of the set

$$\mathbf{X_a} = \bigoplus_{j=0}^{\infty} \mathbf{X}_{aj} \text{ where } \mathbf{X}_{aj} = \bigoplus_{k=1}^{\dim[R(\lambda_j]} \mathbf{X}_{ajk}.$$

Theorem 5. *[2], [12], [13], [16]. If the hypothesis* (H_{a1}) *and* (H_{a2}) *are satisfied, then the equilibrium solution* (U_a^*, V_a^*) *of* (9) *is locally asymptotically stable if* $D_{a1} > D_{a2}$.

Proof. By linearising of the system (9) at the non trivial equilibrium (U_a^*, V_a^*), we have

$$\mathbf{U}_{at} = D_a \triangle \mathbf{U}_a + \mathbf{\Sigma_{U_a}}(U_a^*)\mathbf{U}_a, \quad \text{where} \quad D_a = \begin{pmatrix} D_{a1} & 0 \\ 0 & D_{a2} \end{pmatrix}.$$

From the previous notation, $\mathbf{X_{aj}}$ is invariant set under the operator $D_a\Delta + \mathbf{\Sigma_{U_a}}(U_a^*)$ and μ is an eigenvalue of this operator on $\mathbf{X_{aj}}$, if and only if it is an eigenvalue of $-\lambda_i D_a + \mathbf{\Sigma_{U_a}}(U_a^*)$.The characteristic polynomial of $-\lambda_i D_a + \mathbf{\Sigma_{U_a}}(U_a^*)$ is given by

$$\Phi_i(\mu) = \mu^2 + M_{a1}\mu + M_{a2} \text{ where } M_{a1} = (D_{a1} + D_{a2})\lambda_i - tr(\mathbf{\Sigma_{U_a}}(U_a^*)),$$

$$M_{a2} = D_{a1}D_{a2}\lambda_i^2 + \frac{2D_{a1}2e_2(e_3 - e_2)}{e_3}\lambda_i - \frac{2D_{a2}(2e_2C_a - e_3^2)}{e_3^2}\lambda_i + \det(\mathbf{\Sigma_{U_a}}(U_a^*)).$$

Under the hypothesis (H_{a1}) and (H_{a2}), it is easy to verify that M_{a1} and M_{a2} are all positive. Then it follows from the Routh-Hurwitz stability criterion that, for each $j > 0$, the two roots $\mu_1(\lambda_j), \mu_2(\lambda_j)$ of $\Phi_i(\mu) = 0$ all have negative real parts. ◀

3. 3. Complex model with self and cross diffusion analysis

It is important to see that if only the self-diffusion introduced to the ODE system (7) and the self-diffusion coefficients satisfy $D_{a1} > D_{a2}$, when other conditions are unchanging, the positive equilibrium solution is also locally stable. Now we consider the effect of self and cross diffusion describe in model (8). Indeed, we have the following results. We assume the following hypothesis:

$$(H_{a3}) : \quad \frac{1}{e_3} max\{e_2, C_a\} < 1 \quad \text{and} \quad \sqrt{\frac{C_a^2}{4} + 4e_2C_a} - \frac{C_a}{2} > e_3.$$

Theorem 6. *[2], [14]. Under the hypothesis* (H_{a2}) *and* (H_{a3}), *if* $\lambda_2 < \overline{\lambda}$ *where* λ_2 *is define in the previous notation and* $\overline{\lambda}$ *will be given in* (12), *then there exists a positive* r_0 *such that when* $D_3 > r_0$, *the interior equilibrium* (U_a^*, V_a^*) *of* (8) *is unstable.*

Proof. Let us consider $\Phi(U_a) = (D_{a1}U_a, D_{a2}(1 + D_{a3}U_a)V_a)^T$. By linearising the system (8) at (U_a^*, V_a^*), we obtain $U_{at} = (\Phi_{U_a}(U_a^*)\Delta + \Sigma_{U_a}(U_a^*))U_a$, where

$$\mathbf{\Phi}_{\mathbf{U_a}}(U_a^*) = \begin{pmatrix} D_{a1} & 0 \\ D_{a2}D_{a3}V_a^* & D_{a2} + D_{a2}D_{a3}U_a^* \end{pmatrix}.$$

By some calculations, the characteristic polynomial of

$$-\lambda_i \mathbf{\Phi}_{\mathbf{U_a}}(U_a^*) + \mathbf{\Sigma}_{\mathbf{U_a}}(\mathbf{U_a^*}) \quad \text{is written by} \quad \Upsilon(\mu) = \mu^2 + S_1\mu + S_2, \text{where}$$

$$S_1(\lambda_i) = (D_{a1} + D_{a2} + D_{a2}D_{a3}\frac{e_3 - C_a}{e_3}\lambda_i - \operatorname{tr}(\mathbf{\Sigma}_{\mathbf{U_a}}(\mathbf{U_a^*})) \quad \text{and}$$

$$S_2(\lambda_i) = R_1\lambda_i^2 + R_2\lambda_i + \det(\mathbf{\Sigma}_{\mathbf{U_a}}(\mathbf{U_a^*})),$$

$$R_1 = \frac{2e_2e_3^2D_{a1}(e_3 - e_2) - e_3D_{a2}(2e_2C_a - e_3^2) + D_{a1}D_{a3}(e_3 - C_a)(e_3C_a - 4e_2C_a + e_3^2)}{e_3^2}$$

and

$$R_2 = \frac{e_3D_{a1}D_{a2} + D_{a1}D_{a2}D_{a3}(e_3 - C_a)}{e_3}.$$

Let $\mu_1(\lambda_i), \mu_2(\lambda_i)$ be the two roots of $\Upsilon_i(\mu) = 0$, then $\mu_1\mu_2 = S_2(\mu_i)$. It is obvious that $\mu_1 + \mu_2 < 0$ under the assumption (H_{a2}). In order to have at least one $\operatorname{Re}(\mu_j(\lambda_i)) > 0; j = 1, 2$, it is sufficient that $S_2(\mu_i) < 0$. The next task is to find out the conditions such that $S_2(\mu_i) < 0$. Let $G_a(\lambda) = R_2\lambda^2 + R_1\lambda + \det(\mathbf{\Sigma}_{\mathbf{U_a}}(\mathbf{U_a^*}))$. Let $\overline{\lambda}_1$ and $\overline{\lambda}_2$ be the two roots of $G_a(\lambda) = 0$ with $\operatorname{Re}(\overline{\lambda}_1) \leq \operatorname{Re}(\overline{\lambda}_1)$. Consider the following limits

$$\lim_{D_{a3}\longrightarrow\infty} \frac{R_2}{D_{a3}} = \frac{D_{a1}D_{a2}(e_3 - C_a)}{e_3} \triangleq q_2 > 0$$

and

$$\lim_{D_{a3}\longrightarrow\infty} \frac{R_1}{D_{a3}} = \frac{D_{a2}(e_3 - C_a)(e_3^2 + e_3C_a - 4e_2C_a)}{e_3^3} \triangleq q_1.$$

Since the hypothesis (H_{a3}) hold, we can assert $e_3^2 + e_3C_a - 4e_2C_a < 0$. Using the continuity argument shows that, when D_{a3} is large enough,

$$\begin{aligned} &\lim_{D_{a3}\longrightarrow\infty} \frac{G_a(\lambda)}{D_{a3}} = q_1\lambda + q_2\lambda^2 \quad \text{and so} \quad \lim_{D_{a3}\longrightarrow\infty} \overline{\lambda}_1 = 0, \\ &\lim_{D_{a3}\longrightarrow\infty} \overline{\lambda}_2 = -\tfrac{q_1}{q_2} \triangleq \overline{\lambda} > 0. \end{aligned} \tag{12}$$

Hence there exists a positive number r_0 such that, when $D_{a3} > r_0$, the following holds $G_a(\lambda) < 0$, when we have $\lambda \in (\overline{\lambda}_1, \overline{\lambda}_2)$. Since that $0 < \overline{\lambda}_1 < \overline{\lambda}$, we have $\lambda \in (\overline{\lambda}_1, \overline{\lambda}_2)$. It follows that $S_2(\mu_i) < 0$. ◀

4. Numerical Simulations

In this section, we present a sequence of numerical simulations of the ODE system (2) and the PDE system (6) to support our mathematical results. Hear we analyze the effect of intra-specific predator competition for prey consumption on the dynamics and the effect of cross and self diffusion in the two species. The parameter values used are given in the Table 1 below.

Parameters	Values	References
r_p	2.8	estimated
δ_a	0.62	estimated
K_p	142	estimated
s	84	estimated
β	0.9	estimated
α	0.941	estimated
q_{af}	1	estimated
T_0^p	200	estimated
T_0^f	120	estimated

Table 1. *Numerical values of fish zooplankton model parameters*

4. 1. Numerical results of the ODE model

The numerical simulations are obtain by using MATLAB code for solving and drawing the phase portrait. We observe the behaviour of the solutions of the system (2) by plotting the densities of two species as a function of time if we vary the fishing parameter E_{af}. We observe from these figure that the equilibrium E_{a2} is locally asymptotically stable, if we increase the value of $E_{af} = 0.4$ this equilibrium loses its stability and becomes unstable. Concerning the first numerical experiment, we have the following results:

(a) From Figure $2-(A1)$, it is observed that the system (2) has a unique equilibrium point $(5.16, 1.74)$ which is an asymptotic stable point. We see in the Figure $2-(A1)$ two boundary saddle points $(0,0)$ and $(8.23, 0)$.
(b) At the Figure $2-(A2)$, it is observed that the system (2) has a equilibrium point $(1.38, 0)$ and the saddle points $(0,0)$.
(c) From Figure $2-(A3)$, it is observed that the system (2) has a unique equilibrium point $(2.53, 1.02)$ which is an asymptotic stable point. We see in the Figure $2-(A3)$ two boundary saddle points $(0,0)$ and $(8.21, 0)$.
(d) The system (2) is $2-(A4)$ phase portrait for $E_{af} = 0.25$ approaches to nodal sink $(3.71, 1.49)$. $2-(A5)$ phase portrait for $E_{af} = 0.27$ approaches to spiral sink $(2.37, 0.92)$. $2-(A6)$ trajectory approaches to small limit cycle around the source point $(2.3, 0.87)$ for $E_{af} = 0.39$.

The behaviour of the solutions of the system (2) by plotting the densities of two species as a function of time if we vary the fishing parameter E_{af} is presented in the Figure 3.

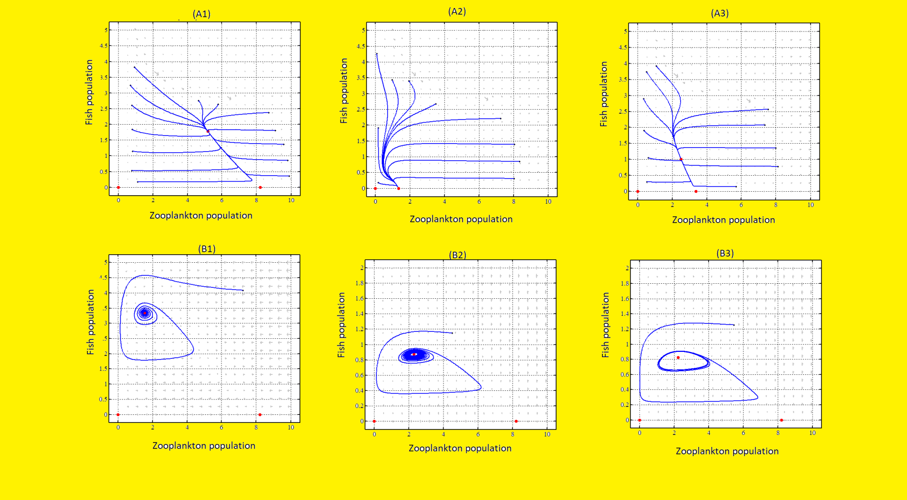

Fig. 2. Dynamics behaviour of the ordinaries differential equation (2) **by using the model parameters give in the Table 1 or plot** $T_{max} = 300$

The parameter values are given in table 1. We observe from these figure that the equilibrium E_{a2} is locally asymptotically stable, if we increase the value of $E_{af} = 0.4$ this equilibrium loses its stability and becomes unstable. Concerning the first numerical experiment, we have the following results:

(a) From Figure $3-(A4)$, it is observed that the system (2) admits a unique equilibrium point $(6.35, 1.04$ which is an asymptotic stable point. We see in the Figure $3-(A4)$ two boundary saddle points $(0, 0)$ and $(8.23, 0)$.

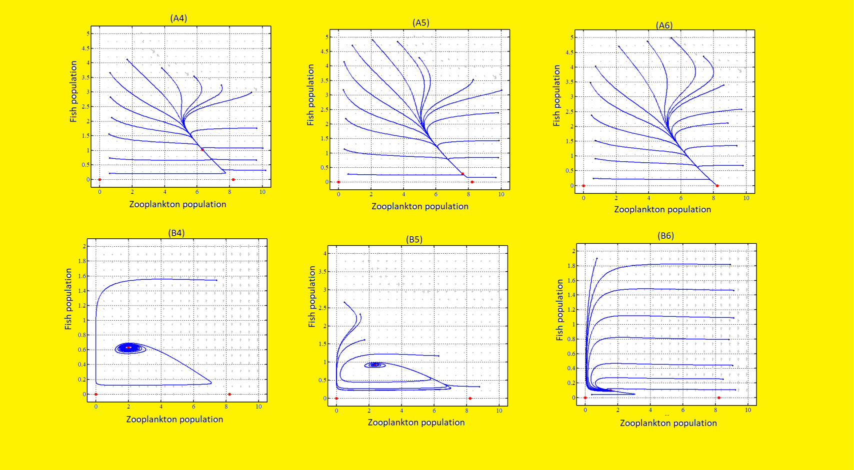

Fig. 3. Dynamics behavior of the ordinaries differential equation (2) **by using the model parameters give in the Table 1 obtained for plot** $T_{max} = 400$

(b) At the Figure $3-(A5)$, it is observed that the system (2) has a non trivial equilibrium point $(7.7, 0.3)$ and the saddle points $(0, 0)$.

(c) From Figure $3-(A6)$, it is observed that the system (2) has two boundary saddle points $(0, 0)$ and $(8.21, 0)$.

(d) The system (2) is $3-(B4)$ phase portrait for $E_{af} = 0.27$ approaches to spiral sink $(2.41, 0.88)$. $3-(B5)$ trajectory approaches to spiral sink $(2.29, 0.97)$ for $E_{af} = 0.46$. $3-(B6)$ phase portrait for $E_{af} = 0.48$ approaches to $(1.58, 0.08)$.

4. 2. Numerical results of the PDE model

The system (6) is studied numerically on a spatial domain $\Omega = [0, 60]\times[0, 60]$ by satisfying the criterion of CFL (Courant-Friedrichs-Levy), [3], [6], [7], [9], [13]. The simulation environment used is the *FreeFem ++ software*. The main objective of these numerical experiments is to understand the fishing effect which are the main factors influencing the stability of the zooplankton fish system. To do this, we will vary some values of the fishing effort coefficients E_{af}, in accordance with the mathematical results establish in the section 3.

In the first plotting, we observe the pattern formation in the spatial domain Ω under the hypotheses $(H_{a1}) - (H_{a3})$. The qualitative results of different pattern formations due to the variation of $Tmax$, are shown in the Figures 4 and 5 by using respectively $E_{af} = 0.02$ and $E_{af} = 0.05$.

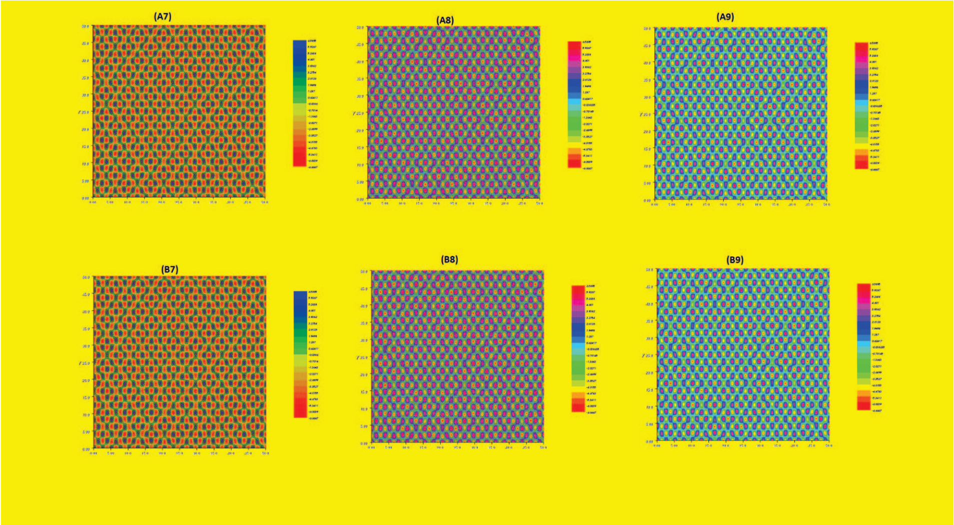

Fig. 4. Dynamics behaviour of two dimensional space: zooplankton [first line] and fish [second line] population density (6). The diffusion coefficient values are $\delta_{a1} = 0.5, \quad \delta_{a2} = 0.5 \quad \delta_{a3} = 0$. The spatial distribution of the system are obtained for plot $t = 400, E_{af} = 0.35$ at $(A13; B13)$, $t = 500, E_{af} = 0.35$ at $(A14; B14)$ and $t = 600, E_{af} = 0.35$ at $(A15; B15)$

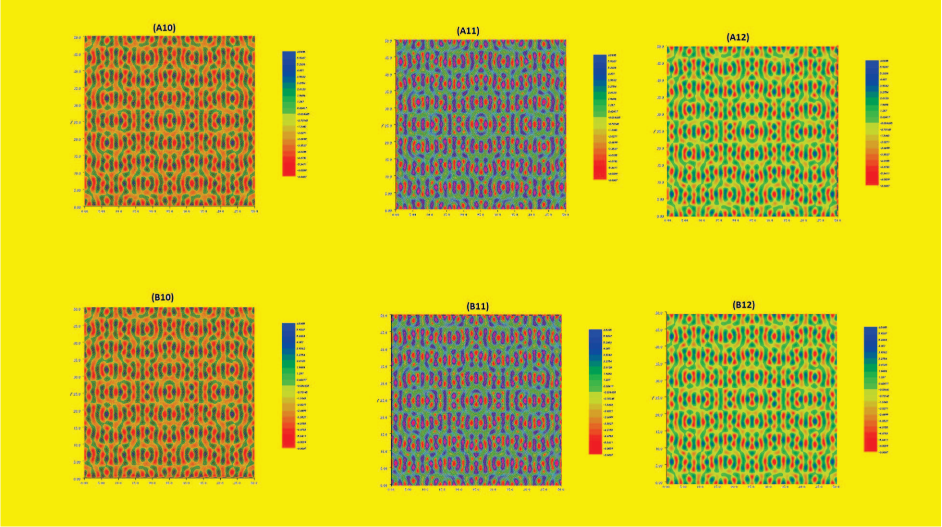

Fig. 5. Dynamics behaviour of two dimensional space: zooplankton [first line] and fish [second line] population density (6)**. The diffusion coefficient values are** $\delta_{a1} = 0.5, \quad \delta_{a2} = 0.5 \quad \delta_{a3} = 0$. **The spatial distribution of the system are obtained for plot** $t = 700, E_f = 0.48$ **at** $(A16; B16)$, $t = 800, E_{af} = 0.48$ **at** $(A17; B17)$ **and** $t = 900, E_{af} = 0.48$ **at** $(A18; B18)$

Remark 2. In biological point of view, these results (Figures 4 - 5 show that there are coexistence between the two populations despite the fishing effect. This means that despite the fishing effect, the fish population persist in the aquatic environment.

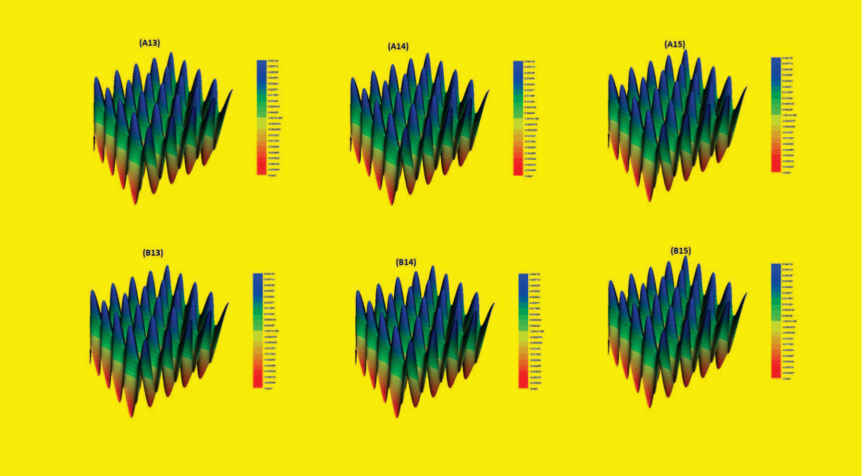

Fig. 6. Dynamics behaviour of two dimensional space: zooplankton [first line] and fish [second line] population density (6)**. The diffusion coefficient values are** $\delta_{a1} = 1.8, \quad \delta_{a2} = 1.8 \quad \delta_{a3} = 6$. **The spatial distribution of the system are obtained for plot** $t = 2000, E_{af} = 0.43$ **at** $(A19; B19)$, $t = 3000, E_{af} = 0.43$ **at** $(A20; B20)$ **and** $t = 4000, E_{af} = 0.43$ **at** $(A21; B21)$

In the second numerical analysis, we observe the dynamics behaviour by considering that the system involve in the self diffusion condition. So we consider that $\delta_{a1} = \delta_{a2} = 0.2$ and $\delta_{a3} = 0$. We plot for $E_{af} = 0.35$ in the Figure 6 and $E_{af} = 0.48$ in the Figure 7 The numerical experiments show that after a transitional phase, the equilibrium can be established with coexistence of the zooplankton fish system. Figures 6 - 7 show the behaviour of the two populations. As a biological view we can say that if the fishing is prated below this value the impact is not significant on the fish population (Figures 6 - 7).

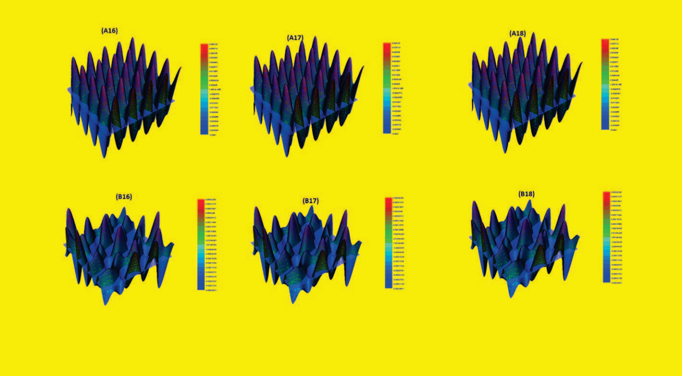

Fig. 7. Dynamics behaviour of two dimensional space: zooplankton [first line] and fish [second line] population density (6)**. The diffusion coefficient values are** $\delta_{a1} = 0.5, \quad \delta_{a2} = 0.5 \quad \delta_{a3} = 6$. **The spatial distribution of the system are obtained for plot** $t = 5000, E_{af} = 0.55$ **at** $(A22; B22)$, $t = 6000, E_{af} = 0.55$ **at** $(A23; B23)$ **and** $t = 7000, E_{af} = 0.55$ **at** $(A24; B24)$

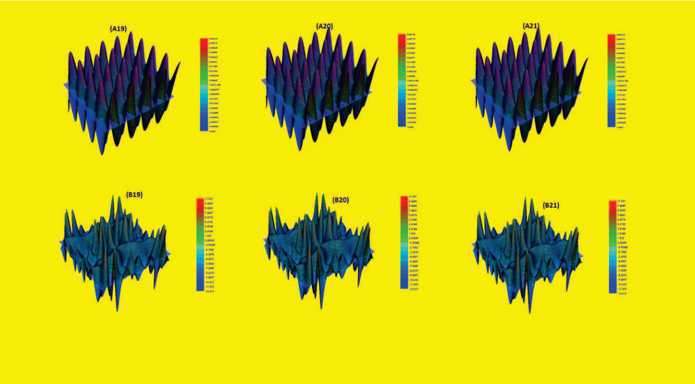

Fig. 8. Dynamics behaviour of two dimensional space: zooplankton [first line] and fish [second line] population density (6)**. The diffusion coefficient values are** $\delta_{a1} = 1.8, \quad \delta_{a2} = 1.8 \quad \delta_{a3} = 6$. **The spatial distribution of the system are obtained for plot** $t = 2000, E_{af} = 0.43$ **at** $(A19; B19)$, $t = 3000, E_{af} = 0.43$ **at** $(A20; B20)$ **and** $t = 4000, E_{af} = 0.43$ **at** $(A21; B21)$

In the last numerical analysis, we observe the dynamics behaviour by considering that the system involve in the self and cross diffusion condition. So we consider that $\delta_{a1} = \delta_{a2} = 1.8$ and $\delta_{a3} = 6$. We plot for $E_{af} = 0.43$ in the Figure 8 and $E_{af} = 0.55$ in the Figure 9 The numerical experiments show that after a transitional phase, the equilibrium can be established with coexistence of the zooplankton fish system. Figures 8 - 9 show the behaviour of the two populations.

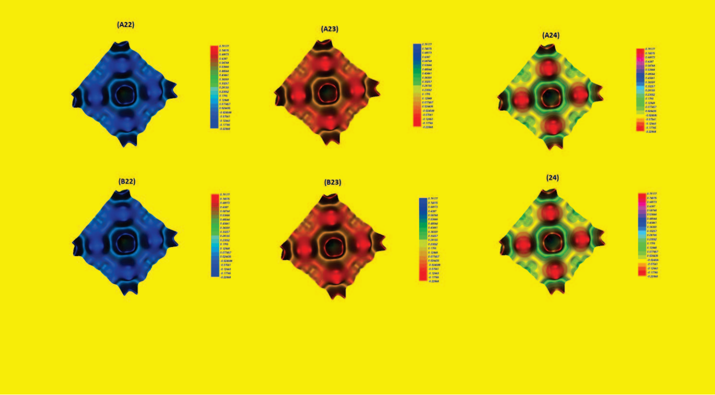

Fig. 9. Dynamics behaviour of two dimensional space: zooplankton [first line] and fish [second line] population density (6)**. The diffusion coefficient values are** $\delta_{a1} = 0.5, \quad \delta_{a2} = 0.5 \quad \delta_{a3} = 6$. **The spatial distribution of the system are obtained for plot** $t = 5000, E_{af} = 0.55$ **at** $(A22; B22)$, $t = 6000, E_{af} = 0.55$ **at** $(A23; B23)$ **and** $t = 7000, E_{af} = 0.55$ **at** $(A24; B24)$

Remark 3. Numerical experiments investigates the influence of varying the parameter E_{af} on the dynamic behaviour of the system (6). It is observed that the behaviour of system (6) is stable with $E_{af} \leq 0.5$.

5. Conclusion

In this paper, we have developed a mathematical theoretical framework for studying two phenomenon: pattern formation and fishing effect in a reaction-diffusion system with cross and self diffusion. By using the stability analysis theories and suitable numerical experiments, we investigate the associated pattern type and a mechanism for pattern selection with fishing effect. The proposed approach has applicability to other reaction-diffusion systems including cross-diffusion, such as zooplankton and food chain models. It is of great interest to us the development of a general mathematical and numerical framework that allows for the treatment of certain degenerate quasilinear parabolic systems modelling several important phenomena in halieutic area. The spatial patterns appear in the PDE systems with cross-diffusion when the effect of fishing effort is larger than a given critical value $E_{max} = 0.5$. The selection of the complex spatial pattern transform from stripes to spots when we consider certain values of $\delta_{a1} = 0.5, \delta_{a2} = 0.5$ and $\delta_{a3} = 6$. It is well-known that for the reaction diffusion system, the formation of patterns of the two species does not occur in certain assumption concerning the critical parameter E_{max}.

Acknowledgements. We would like to thank the referees for their careful reading and their useful remarks.

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