Baku Mathematical Journal
202X, Vol. XX, No X, PP. XX-XX
https://doi.org/10.32010/j.bmj.202X.XX

SOME PROPERTIES OF ONE CLASS OF VECTOR POTENTIALS WITH WEAK SINGULARITIES

V.O. SAFAROVA

Received: date / Revised: date / Accepted: date

Abstract. *The paper shows the boundedness and compactness of the operator generated by a vector potential with a weak singularity in generalized Hölder spaces.*

Keywords: Helmholtz equation, vector potential, generalized Hölder space

Mathematics Subject Classification (2020): 45E05, 31B10

1. Introduction and Problem Statement

It is known that (see [1, pp. 153-154]) the internal and external electrical boundary-value problem is reduced to a system of integral equations depending on the potential vector with a weak singularity

$$(Af)(x) = 2\int_{\Omega} \Phi_k(x,y)\ [n(x),[n(y),f(y)]]\, d\Omega_y,\quad x=(x_1,x_2,x_3)\in\Omega, \tag{1}$$

where $\Omega \subset R^3$ is the Lyapunov surface with the exponent $0 < \alpha \leq 1$; $n(x) = (n_1(x), n_2(x), n_3(x))$ is the outer unit normal at the point $x \in \Omega$; the notation $[a,b]$ means the vector product of vectors a and b; $f(x) = (f_1(x), f_2(x), f_3(x))$ is the vector function continuous on the surface Ω,

$$\Phi_k(x,y) = \frac{\exp(ik|x-y|)}{4\pi|x-y|},\quad x,y\in R^3,\quad x\neq y,$$

fundamental solution of the Helmholtz equation $\Delta u + k^2 u = 0$, Δ is the Laplace operator, and k is a wave number with $Im\, k \geq 0$.

Note that some properties of the operator generated by the derivative of the logarithmic potential of the double layer were studied in [6]; some properties of the operator generated by the derivative of the acoustic potential of a simple layer were studied in [5]; and some properties of the operator generated by the normal derivative of the acoustic

Vafa O. Safarova
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
E-mail: vefa-seferova-91@bk.ru

potential of a double layer in generalized Hölder spaces were studied in [4]. It should be noted that in [1, pp. 73, 154] it is shown that if Ω is a closed and twice continuously differentiable surface in R^3, then the operator A works boundedly from the space of continuous functions onto the Hölder space and is compact in the space of continuous functions, as well as in the Hölder space. Here we prove the validity of the Zygmund type estimate and study some properties of the operator generated by the vector potential (1) in generalized Hölder spaces.

2. Estimate of A. Zygmund Type Estimate for the Operator Generated by the Vector Potential (1)

Let us introduce the modulus of continuity of a vector function$f \in C\left(\Omega\right)$:

$$\bar{\omega}_f\left(\delta\right) = \max_{\substack{|x-y|\leq\delta \\ x,\, y\in\Omega}} \left|f\left(x\right) - f\left(y\right)\right|, \delta > 0,$$

where

$$\left|f\left(x\right) - f\left(y\right)\right| = \sqrt{\left(f_1\left(x\right) - f_1\left(y\right)\right)^2 + \left(f_2\left(x\right) - f_2\left(y\right)\right)^2 + \left(f_3\left(x\right) - f_3\left(y\right)\right)^2},$$

and $C\left(\Omega\right)$ denotes the space of all continuous functions on the surface Ω with the norm $\|f\|_\infty = \max_{x\in\Omega} \left|f\left(x\right)\right|$. Since the function $\bar{\omega}_f\left(\delta\right)$ does not have the property of semi-additivity, the function "corrected" by S.B. Stechkin is chosen as the main characteristic

$$\omega_f\left(\delta\right) = \delta \sup_{\tau\geq\delta} \frac{\bar{\omega}_f\left(\tau\right)}{\tau}, \delta > 0.$$

It is known that the function $\omega_f\left(\delta\right)$ has the following properties: $\omega_f\left(\delta\right)$ is non-negative, semi-additive, and non-decreasing; the function $\omega_f\left(\delta\right)/\delta$ is not increasing; $\lim_{\delta\to0} \omega_f\left(\delta\right) = 0$ and $\omega_f\left(C\delta\right) \leq \left(1 + C\right)\omega_f\left(\delta\right)$, where $C = const > 0$.

Let us denote by $d > 0$ the radius of the standard sphere for Ω (see [7, p. 400]) and let $\Omega_\varepsilon\left(x\right) = \left\{y \in \Omega : \ |x - y| < \varepsilon\right\}$, where $x \in \Omega$ and $\varepsilon > 0$. It is known that for each $x \in \Omega$ the set $\Omega_d\left(x\right)$ is projected uniquely onto the set $\Pi_d(x)$ lying in the tangent plane $\Gamma(x)$ to Ω at the point x. On the piece $\Omega_d(x)$ we choose a local rectangular coordinate system (u, v, w), with the origin at the point x, where the w-axis is directed along the normal $n\left(x\right)$, and the u- and v-axes lie in the tangent plane $\Gamma\left(x\right)$. Then in these coordinates the neighborhood $\Omega_d(x)$ can be defined by the equation $w = \psi\left(u,\, v\right),\ \left(u,\, v\right) \in \Pi_d(x)$, where $\psi \in H_{1,\alpha}\left(\Pi_d\left(x\right)\right)$ and $\psi\left(0,\, 0\right) = 0$, $\frac{\partial\psi(0,\,0)}{\partial u} = 0$, $\frac{\partial\psi(0,\,0)}{\partial v} = 0$. Here $H_{1,\alpha}\left(\Pi_d\left(x\right)\right)$ denotes the linear space of all continuously differentiable functions ψ on $\Pi_d\left(x\right)$, whose $grad\psi$ satisfies the Holder condition with the exponent $0 < \alpha \leq 1$, i.e.,

$$\left|grad\psi\left(u_1, v_1\right) - grad\,\psi\left(u_2, v_2\right)\right| \leq$$

$$\leq M_\psi \left(\sqrt{\left(u_1 - u_2\right)^2 + \left(v_1 - v_2\right)^2}\right)^\alpha, \forall\left(u_1, v_1\right), \left(u_2, v_2\right) \in \Pi_d(x),$$

where M_ψ is a positive constant depending on ψ, and not from (u_1, v_1) or (u_2, v_2).

Theorem 1. *Let $\Omega \subset R^3$ be the Lyapunov surface with the exponent $0 < \alpha \leq 1$ and vector functionf $(x) = (f_1(x), f_2(x), f_3(x))$ be continuous on the surface Ω. Then for any points $x', x'' \in \Omega$, the following estimates are valid*

$$|(Af)(x') - (Af)(x'')| \leq M_f \left(h^\alpha + \int\limits_0^h \omega_f(r)\,dr + h \int\limits_h^d \frac{\omega_f(r)}{r} dr \right) \quad for \quad 0 < \alpha < 1,$$

$$|(Af)(x') - (Af)(x'')| \leq M_f \left(h\,|\ln h| + \int\limits_0^h \omega_f(r)\,dr + h \int\limits_h^d \frac{\omega_f(r)}{r} dr \right) \quad for \quad \alpha = 1,$$

where $h = |\,x' - x''\,|$ and M_f is a positive constant depending only on Ω, k and f.

Proof. Let's take any points $x',\ x'' \in \Omega$ such that the value h is less than $d/2$. It is not difficult to see that

$$(Af)(x) = e_1 ((A_{11}f)(x) + (A_{12}f)(x)) +$$

$$+ e_2 ((A_{21}f)(x) + (A_{22}f)(x)) + e_3 ((A_{31}f)(x) + (A_{32}f)(x)),$$

where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$,

$$(A_{11}f)(x) = 2n_2(x) \int\limits_\Omega \Phi_k(x,\, y)\,(n_1(y) f_2(y) - n_2(y) f_1(y))\, d\Omega_y,$$

$$(A_{12}f)(x) = 2n_3(x) \int\limits_\Omega \Phi_k(x,\, y)\,(n_1(y) f_3(y) - n_3(y) f_1(y))\, d\Omega_y,$$

$$(A_{21}f)(x) = 2n_3(x) \int\limits_\Omega \Phi_k(x,\, y)\,(n_2(y) f_3(y) - n_3(y) f_2(y))\, d\Omega_y,$$

$$(A_{22}f)(x) = 2n_1(x) \int\limits_\Omega \Phi_k(x,\, y)\,(n_2(y) f_1(y) - n_1(y) f_2(y))\, d\Omega_y,$$

$$(A_{31}f)(x) = 2n_1(x) \int\limits_\Omega \Phi_k(x,\, y)\,(n_3(y) f_1(y) - n_1(y) f_3(y))\, d\Omega_y,$$

and

$$(A_{32}f)(x) = 2n_2(x) \int\limits_\Omega \Phi_k(x,\, y)\,(n_3(y) f_2(y) - n_2(y) f_3(y))\, d\Omega_y.$$

It is not difficult to show that the following representation is correct:

$$(A_{11}f)(x') - (A_{11}f)(x'') =$$

$$= 2\left(n_2\left(x'\right) - n_2\left(x''\right)\right) \int\limits_{\Omega\backslash\Omega_d(x')} \Phi_k\left(x',\, y\right)\left(n_1\left(y\right) f_2\left(y\right) - n_2\left(y\right) f_1\left(y\right)\right) d\Omega_y +$$

$$+ 2 n_2\left(x''\right) \int\limits_{\Omega\backslash\Omega_d(x')} \left(\Phi_k\left(x',\, y\right) - \Phi_k\left(x'',\, y\right)\right)\left(n_1\left(y\right) f_2\left(y\right) - n_2\left(y\right) f_1\left(y\right)\right) d\Omega_y +$$

$$+ 2 n_2\left(x'\right) \int\limits_{\Omega_{h/2}(x') \bigcup \Omega_{h/2}(x'')} \Phi_k\left(x',\, y\right)\left(\left(n_1\left(y\right) f_2\left(y\right) - n_2\left(y\right) f_1\left(y\right)\right) -\right.$$

$$\left. - \left(n_1\left(x'\right) f_2\left(x'\right) - n_2\left(x'\right) f_1\left(x'\right)\right)\right) d\Omega_y -$$

$$- 2 n_2\left(x''\right) \int\limits_{\Omega_{h/2}(x') \bigcup \Omega_{h/2}(x'')} \Phi_k\left(x'',\, y\right)\left(\left(n_1\left(y\right) f_2\left(y\right) - n_2\left(y\right) f_1\left(y\right)\right) -\right.$$

$$\left. - \left(n_1\left(x''\right) f_2\left(x''\right) - n_2\left(x''\right) f_1\left(x''\right)\right)\right) d\Omega_y +$$

$$+ 2 \int\limits_{\Omega_d(x')\backslash\left(\Omega_{h/2}(x') \bigcup \Omega_{h/2}(x'')\right)} \left(\left(n_1\left(y\right) f_2\left(y\right) - n_2\left(y\right) f_1\left(y\right)\right) - \left(n_1\left(x'\right) f_2\left(x'\right) - n_2\left(x'\right) f_1\left(x'\right)\right)\right) \times$$

$$\times \left(n_2\left(x'\right) \Phi_k\left(x',\, y\right) - n_2\left(x''\right) \Phi_k\left(x'',\, y\right)\right) d\Omega_y +$$

$$+ 2\left(\left(n_1\left(x'\right) f_2\left(x'\right) - n_2\left(x'\right) f_1\left(x'\right)\right) - \left(n_1\left(x''\right) f_2\left(x''\right) - n_2\left(x''\right) f_1\left(x''\right)\right)\right) \times$$

$$\times n_2\left(x''\right) \int\limits_{\Omega_{h/2}(x') \bigcup \Omega_{h/2}(x'')} \Phi_k\left(x'',\, y\right) d\Omega_y +$$

$$+ 2\left(n_1\left(x'\right) f_2\left(x'\right) - n_2\left(x'\right) f_1\left(x'\right)\right) \int\limits_{\Omega_d(x')} \left(n_2\left(x'\right) \Phi_k\left(x', y\right) - n_2\left(x''\right) \Phi_k\left(x'', y\right)\right) d\Omega_y. \quad (2)$$

We denote the terms on the right-hand side of equality (2) by K_1, K_2, K_3, K_4, K_5, K_6, and K_7, respectively.

According to the definition of a Lyapunov surface with the exponent $0 < \alpha \leq 1$,

$$\left|n\left(x\right) - n\left(y\right)\right| \leq M^1 \left|x - y\right|^{\alpha}, \quad \forall x, y \in \Omega. \quad (3)$$

Then, considering that

$$P\left(x\right) = \int\limits_{\Omega\backslash\Omega_d(x')} \Phi_k\left(x,\, y\right)\left(n_1\left(y\right) f_2\left(y\right) - n_2\left(y\right) f_1\left(y\right)\right) d\Omega_y, x \in \Omega,$$

is a proper integral, we have:

$$\left|K_1\right| \leq M \left\|f\right\|_{\infty} h^{\alpha}.$$

It is obvious that the function $P\left(x\right)$ is continuously differentiable on the surface Ω. Then

$$\left|K_2\right| = 2\left|n_2\left(x''\right)\left(P\left(x'\right) - P\left(x''\right)\right)\right| \leq M \left\|f\right\|_{\infty} h.$$

[1] Here and below, M will denote positive constants that are distinct in different inequalities.

Let us represent the expression K_3 in the form $K_3 = 2n_2(x')(K_3' + K_3'')$, where

$$K_3' = \int\limits_{\Omega_{h/2}(x')} \Phi_k(x',\, y)\,((n_1(y)\, f_2(y) - n_2(y)\, f_1(y)) - (n_1(x')\, f_2(x') - n_2(x')\, f_1(x')))\, d\Omega_y$$

and

$$K_3'' = \int\limits_{\Omega_{h/2}(x'')} \Phi_k(x',\, y)\,((n_1(y)\, f_2(y) - n_2(y)\, f_1(y)) - (n_1(x')\, f_2(x') - n_2(x')\, f_1(x')))\, d\Omega_y.$$

Since

$$\begin{aligned}(n_1(y)\, f_2(y) - n_2(y)\, f_1(y)) - (n_1(x')\, f_2(x') - n_2(x')\, f_1(x')) =\\ = (n_1(y) - n_1(x'))\, f_2(y) + (f_2(y) - f_2(x'))\, n_1(x') +\\ + (n_2(x') - n_2(y))\, f_1(x') + (f_1(x') - f_1(y))\, n_2(y)\,,\end{aligned} \tag{4}$$

then taking into account inequality (3) and the continuity of the vector function $f(x)$ on the surface Ω, we obtain that for any $y \in \Omega_{h/2}(x')$ the following inequality is valid

$$\begin{aligned}|(n_1(y)\, f_2(y) - n_2(y)\, f_1(y)) - (n_1(x')\, f_2(x') - n_2(x')\, f_1(x'))| \le\\ \le M\left(\|f\|_\infty |y - x'|^\alpha + \omega_f(|y - x'|)\right).\end{aligned}$$

Therefore, using the formula for reducing a surface integral to a double integral (see [2, p. 276]) and moving to the polar coordinate system, we find:

$$|K_3'| \le M \int\limits_{\Omega_{h/2}(x')} \frac{\|f\|_\infty |y - x'|^\alpha + \omega_f(|y - x'|)}{|y - x'|} d\Omega_y =$$

$$= M \int\limits_{\Pi_{h/2}(x')} \frac{\|f\|_\infty \left(\sqrt{u^2 + v^2}\right)^\alpha + \omega_f\left(\sqrt{u^2 + v^2}\right)}{\sqrt{u^2 + v^2}} \times$$

$$\times \sqrt{1 + \left(\frac{\partial\psi(u,\, v)}{\partial u}\right)^2 + \left(\frac{\partial\psi(u,\, v)}{\partial v}\right)^2} du\, dv \le$$

$$\le M \iint_{u^2 + v^2 \le (h/2)^2} \frac{\|f\|_\infty \left(\sqrt{u^2 + v^2}\right)^\alpha + \omega_f\left(\sqrt{u^2 + v^2}\right)}{\sqrt{u^2 + v^2}} dudv =$$

$$= M \int\limits_0^{2\pi} \int\limits_0^{h/2} \frac{\|f\|_\infty r^\alpha + \omega_f(r)}{r} r dr d\tau \le M\left(\|f\|_\infty h^{1+\alpha} + \int\limits_0^h \omega_f(r)\, dr\right).$$

Moreover, taking into account the expansion (4) and the inequality

$$h/2 \le |y - x'| \le 3h/2, y \in \Omega_{h/2}(x''),$$

we obtain:

$$\left|K_3''\right| \le M \int\limits_{\Omega_{h/2}(x'')} \frac{\|f\|_\infty \left|y-x'\right|^\alpha + \omega_f\left(\left|y-x'\right|\right)}{\left|y-x'\right|} d\Omega_y \le$$

$$\le \frac{M\left(\left(3h/2\right)^\alpha \|f\|_\infty + \omega_f\left(3h/2\right)\right)}{h/2} \int\limits_{\Omega_{h/2}(x'')} d\Omega_y \le M\left(\|f\|_\infty h^{1+\alpha} + \omega_f\left(h\right)h\right).$$

Since,

$$\int\limits_0^h \omega_f\left(r\right)dr = \int\limits_0^h \frac{\omega_f\left(r\right)}{r} r dr \ge \frac{\omega_f\left(h\right)}{h} \int\limits_0^h r dr = \frac{\omega_f\left(h\right)}{h}\frac{h^2}{2} = \frac{1}{2}\omega_f\left(h\right)h,$$

we get that

$$\left|K_3''\right| \le M\left(\|f\|_\infty h^{1+\alpha} + \int\limits_0^h \omega_f\left(r\right)dr\right).$$

Summing up the obtained estimates for the expressions K_3' and K_3'', we obtain the estimates:

$$\left|K_3\right| \le M\left(\|f\|_\infty h^{1+\alpha} + \int\limits_0^h \omega_f\left(r\right)dr\right). \tag{5}$$

Proceeding in exactly the same way as in the proof of inequality (5), we obtain the validity of the estimate

$$\left|K_4\right| \le M\left(\|f\|_\infty h^{1+\alpha} + \int\limits_0^h \omega_f\left(r\right)dr\right).$$

Now let us evaluate the expression K_5. Taking into account the inequalities

$$\left|x''-y\right| \ge \frac{1}{2}\left|x'-x''\right|, y \in \Omega_d\left(x'\right) \setminus \left(\Omega_{h/2}\left(x'\right) \cup \Omega_{h/2}\left(x''\right)\right),$$

and

$$\left|x'-y\right| \ge \frac{1}{2}\left|x'-x''\right|, y \in \Omega_d\left(x'\right) \setminus \left(\Omega_{h/2}\left(x'\right) \cup \Omega_{h/2}\left(x''\right)\right),$$

we obtain the validity of the following inequalities:

$$\left|x'-y\right| \le \left|x'-x''\right| + \left|x''-y\right| \le 3\left|x''-y\right|, y \in \Omega_d\left(x'\right) \setminus \left(\Omega_{h/2}\left(x'\right) \cup \Omega_{h/2}\left(x''\right)\right),$$

and

$$\left|x''-y\right| \le 3\left|x'-y\right|, y \in \Omega_d\left(x'\right) \setminus \left(\Omega_{h/2}\left(x'\right) \cup \Omega_{h/2}\left(x''\right)\right).$$

Then for any point $y \in \Omega_d\left(x'\right) \setminus \left(\Omega_{h/2}\left(x'\right) \cup \Omega_{h/2}\left(x''\right)\right)$, we have:

$$\left|n_2\left(x'\right)\Phi_k\left(x', y\right) - n_2\left(x''\right)\Phi_k\left(x'', y\right)\right| =$$

$$= \left| \left(n_2 \left(x' \right) - n_2 \left(x'' \right) \right) \Phi_k \left(x',\, y \right) + \left(\Phi_k \left(x',\, y \right) - \Phi_k \left(x'',\, y \right) \right) n_2 \left(x'' \right) \right| \le$$

$$\le M \left(\frac{\left| x' - x'' \right|^\alpha}{\left| x' - y \right|} + \frac{\left| x' - x'' \right| + \left| x' - y \right| \left| x' - x'' \right|}{\left| x' - y \right|^2} \right) \le M \left(\frac{h^\alpha}{\left| x' - y \right|} + \frac{h}{\left| x' - y \right|^2} \right). \quad (6)$$

As a result, taking into account the expansion (4) and the formula for reducing a surface integral to a double integral, we obtain:

$$\left| K_5 \right| \le M \int\limits_{\Omega_d(x') \backslash \Omega_{h/2}(x')} \left(\left\| f \right\|_\infty \left| y - x' \right|^\alpha + \omega_f \left(\left| y - x' \right| \right) \right) \left(\frac{h^\alpha}{\left| x' - y \right|} + \frac{h}{\left| x' - y \right|^2} \right) d\Omega_y =$$

$$= M \left(\left\| f \right\|_\infty h^\alpha \int\limits_{\Omega_d(x') \backslash \Omega_{h/2}(x')} \frac{d\Omega_y}{\left| x' - y \right|^{1-\alpha}} + \left\| f \right\|_\infty h \int\limits_{\Omega_d(x') \backslash \Omega_{h/2}(x')} \frac{d\Omega_y}{\left| x' - y \right|^{2-\alpha}} + \right.$$

$$\left. + h^\alpha \int\limits_{\Omega_d(x') \backslash \Omega_{h/2}(x')} \frac{\omega_f \left(\left| y - x' \right| \right)}{\left| x' - y \right|} d\Omega_y + h \int\limits_{\Omega_d(x') \backslash \Omega_{h/2}(x')} \frac{\omega_f \left(\left| y - x' \right| \right)}{\left| x' - y \right|^2} d\Omega_y \right) \le$$

$$\le M \left(\left\| f \right\|_\infty h^\alpha + h \int\limits_h^d \frac{\omega_f (r)}{r} dr \right).$$

Substituting $y = x''$ into expansion (4), we find:

$$\left| \left(n_1 \left(x' \right) f_2 \left(x' \right) - n_2 \left(x' \right) f_1 \left(x' \right) \right) - \left(n_1 \left(x'' \right) f_2 \left(x'' \right) - n_2 \left(x'' \right) f_1 \left(x'' \right) \right) \right| \le$$

$$\le M \left(\left\| f \right\|_\infty h^\alpha + \omega_f (h) \right).$$

Then, taking into account the inequality

$$h/2 \le \left| y - x'' \right| \le 3h/2, y \in \Omega_{h/2} \left(x' \right),$$

and using the formula for reducing a surface integral to a double integral, we obtain the following estimate for the expression K_6:

$$\left| K_6 \right| \le M \left(\left\| f \right\|_\infty h^\alpha + \omega_f (h) \right) \left(\int\limits_{\Omega_{h/2}(x')} \frac{d\Omega_y}{\left| y - x'' \right|} + \int\limits_{\Omega_{h/2}(x'')} \frac{d\Omega_y}{\left| y - x'' \right|} \right) \le$$

$$\le M \left(\left\| f \right\|_\infty h^\alpha + \omega_f (h) \right) \left(\frac{2}{h} \int\limits_{\Omega_{h/2}(x')} d\Omega_y + \int\limits_{\Omega_{h/2}(x'')} \frac{d\Omega_y}{\left| y - x'' \right|} \right) \le$$

$$\le M \left(\left\| f \right\|_\infty h^{1+\alpha} + \omega_f (h) \, h \right) \le M \left(\left\| f \right\|_\infty h^{1+\alpha} + \int\limits_0^h \omega_f (r) \, dr \right).$$

From inequality (6), we obtain

$$|K_7| \leq M \|f\|_\infty \int\limits_{\Omega_d(x')} \left(\frac{h^\alpha}{|x'-y|} + \frac{h}{|x'-y|^2} \right) d\Omega_y \leq M \|f\|_\infty \left(h^\alpha + h |\ln h| \right).$$

As a result, summing up the obtained estimates for the expressions K_1, K_2, K_3, K_4, K_5, K_6 and K_7, we find:

$$|(A_{11}f)(x') - (A_{11}f)(x'')| \leq M_f \left(h^\alpha + h|\ln h| + \int\limits_0^h \omega_f(r)\,dr + h \int\limits_h^d \frac{\omega_f(r)}{r} dr \right).$$

As you can see, if $0 < \alpha < 1$, then

$$|(A_{11}f)(x') - (A_{11}f)(x'')| \leq M_f \left(h^\alpha + \int\limits_0^h \omega_f(r)\,dr + h \int\limits_h^d \frac{\omega_f(r)}{r} dr \right),$$

and if $0 < \alpha < 1$, then

$$|(A_{11}f)(x') - (A_{11}f)(x'')| \leq M_f \left(h|\ln h| + \int\limits_0^h \omega_f(r)\,dr + h \int\limits_h^d \frac{\omega_f(r)}{r} dr \right).$$

It is easy to show that similar estimates are also valid for the expressions $(A_{12}f)(x)$, $(A_{21}f)(x)$, $(A_{22}f)(x)$, $(A_{31}f)(x)$, and $(A_{32}f)(x)$. This completes the proof of the theorem. ◀

Theorem 2. *Let $\Omega \subset R^3$ be the Lyapunov surface with the exponent $0 < \alpha \leq 1$ and the vector function $f(x) = (f_1(x), f_2(x), f_3(x))$ be continuous on the surface Ω. Then $Af \in C(\Omega)$ and*

$$\omega_{Af}(h) \leq M_f \left(h^\alpha + \int\limits_0^h \omega_f(r)\,dr + h \int\limits_h^{diam\,\Omega} \frac{\omega_f(r)}{r} dr \right) \quad for \quad 0 < \alpha < 1,$$

$$\omega_{Af}(h) \leq M_f \left(h|\ln h| + \int\limits_0^h \omega_f(r)\,dr + h \int\limits_h^{diam\,\Omega} \frac{\omega_f(r)}{r} dr \right) \quad for \quad \alpha = 1,$$

where $h \in (0, diam\,\Omega]$ and M_f is a positive constant depending only on Ω,k, and f.

Proof. The continuity of the function $(Af)(x)$ on Ω follows directly from Theorem 1. Let

$$Z_0(h) = \begin{cases} h^\alpha + \int\limits_0^h \omega_f(r)\,dr + h \int\limits_h^{diam\,\Omega} \frac{\omega_f(r)}{r} dr \ \ for\ 0 < \alpha < 1, \\ h|\ln h| + \int\limits_0^h \omega_f(r)\,dr + h \int\limits_h^{diam\,\Omega} \frac{\omega_f(r)}{r} dr \ \ for\ \alpha = 1. \end{cases}$$

Applying L'Hôpital's rule, we obtain

$$\lim_{h\to 0} h \int_h^{diam\,\Omega} \frac{\omega_f(r)}{r} dr = \lim_{h\to 0} \frac{\int_h^{diam\,\Omega} \frac{\omega_f(r)}{r} dr}{\frac{1}{h}} = \lim_{h\to 0} \frac{-\frac{\omega_f(h)}{h}}{-\frac{1}{h^2}} = \lim_{h\to 0} h\omega_f(h) = 0,$$

which means that $\lim_{h\to 0} Z_0(h) = 0$.

Since the derivative of the function $Z_0(h)$ is non-negative, i.e., the function $Z_0(h)$ does not decrease, then from Theorem 1 we obtain:

$$\bar{\omega}_{Af}(h) = \max_{\substack{|x'-x''|\le h \\ x',x''\in\Omega}} |(Af)(x') - (Af)(x'')| \le$$

$$\le M_f \max_{\substack{|x'-x''|\le h \\ x',x''\in\Omega}} Z_0(|x'-x''|) = M_f Z_0(h), h \in (0, d/2].$$

Further, taking into account that the derivative of the function $Z_0(h)/h$ is non-positive, i.e., the function $Z_0(h)/h$ does not increase, we have:

$$\omega_{Af}(h) = h \sup_{\tau\ge h} \frac{\bar{\omega}_{Af}(\tau)}{\tau} \le h \sup_{\tau\ge h} \frac{M_f Z_0(\tau)}{\tau} = h\frac{M_f Z_0(h)}{h} = M_f Z_0(h), h \in (0, d/2].$$

Obviously, this estimate is valid for all $h \in (0, diam\,\Omega]$. The theorem has been proven. ◄

3. Some Properties of the Operator Generated by the Vector Potential (1)

We introduce the following class of functions defined on $(0, diam\,\Omega]$:

$$J(\Omega) = \left\{\varphi : \ \varphi\uparrow, \ \lim_{h\to 0}\varphi(h) = 0, \ \varphi(h)/h\downarrow\right\},$$

and let's consider the function

$$Z(h,\varphi) = \begin{cases} h^\alpha + \int_0^h \varphi(r)\,dr + h \int_h^{diam\,\Omega} \frac{\varphi(r)}{r} dr \ \ for\ 0<\alpha<1, \\ h\,|\ln h| + \int_0^h \varphi(r)\,dr + h \int_h^{diam\,\Omega} \frac{\varphi(r)}{r} dr \ \ for\ \alpha = 1, \end{cases}$$

where the sign ↑ means non-decreasing functions, and the sign ↓ means non-increasing functions. Where it will not cause misunderstanding, we will sometimes write $Z(h)$ and $Z(\varphi)$ instead of $Z(h,\varphi)$. In the proof of Theorem 2, it is shown that$Z \in J(\Omega)$.

Let $\varphi \in J(\Omega)$. By $H(\varphi)$ we denote the linear space of all vector functions f defined on the surface Ω and satisfying the condition

$$|f(x) - f(y)| \leq M_f \varphi(|x-y|), x, y \in \Omega,$$

where M_f is a positive constant depending on f, not on the point x or y. As can be seen, if $f \in H(\varphi)$, then the function f is uniformly continuous on Ω. Moreover, it is known that (see [3, p. 60]) the space $H(\varphi)$ is a Banach space with the norm

$$\|f\|_{H(\varphi)} = \|f\|_\infty + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x) - f(y)|}{\varphi(|x-y|)}.$$

The next theorem follows from Theorem 2.

Theorem 3. *Let $\varphi \in J(\Omega)$. Then the operator A acts boundedly from space $H(\varphi)$ onto space $H(Z(\varphi))$.*

Let $\varphi \in J(\Omega)$ and $\lim\limits_{h\to 0} \frac{Z(h,\varphi)}{\varphi(h)} = 0$. It is known that (see [3, p. 70]) the space $H(Z(\varphi))$ is compactly embedded in the space $H(\varphi)$. Then the following theorem is true.

Theorem 4. *Let $\varphi \in J(\Omega)$ and $Z(h,\varphi) = o(\varphi(h))$, $h \to 0$. Then the operator $A : H(\varphi) \to H(\varphi)$ is compact.*

Let us denote by H_β the Hölder space with the exponent $0 < \beta \leq 1$, i.e., the space H_β is the space $H(\varphi)$ at $\varphi(h) = h^\beta$. It is obvious that if $f \in H_\beta$, then

$$Z_0(h) = \begin{cases} h^\alpha \ for \ \ 0 < \alpha < 1, \\ h\,|\ln h| \ \ for \ \ \alpha = 1. \end{cases}$$

Then we obtain the following corollary from Theorems 3 and 4.

Corollary. *The operator A acts boundedly from the space H_β onto the space H_α and is compact in the space H_β for $0 < \beta < \alpha$.*

It should be pointed out that, in particular, if $\alpha = 1$, then from Corollary we obtain the result of the work [1, pp. 73, 154] for the compactness of the operator A in the space H_β.

References

1. Colton D.L., Kress R. *Integral Equation Methods in Scattering Theory.* John Wiley & Sons, New York, 1983; Mir, Moscow, 1987.
2. Fihtengoltz G.M. *The Course of Differential and Integral Calculus*, **3**. Nauka. Moscow, 1969 (in Russian).
3. Guseinov A.I., Muhtarov H.Š. *Introduction to the Theory of Nonlinear Singular Integral Equations.* Nauka, Moscow, 1980 (in Russian).

4. Khalilov E.G. Some properties of the operators generated by a derivative of the acoustic double layer potential. *Sib. Math. J.*, 2014, **55** (3), pp. 564-573.
5. Khalilov E.H. Properties of the operator generated by the derivative of the acoustic single layer potential. *J. Math. Sci.*, 2018, **231** (2), pp. 168-180.
6. Khalilov E.H., Bakhshaliyeva M.N. On the derivative of the double-layer logarithmic potential. *Vestn. Tomsk. Gos. Univ. Mat. Mech.*, 2019, (62), pp. 38-54 (in Russian).
7. Vladimirov V.S. *Equations of Mathematical Physics*. Nauka, Moscow, 1981 (in Russian).