Baku Mathematical Journal
202X, Vol. XX, No X, PP. XX-XX
https://doi.org/10.32010/j.bmj.202X.XX

ASYMPTOTIC EXPANSION FOR THE THIRD ORDER MOMENT OF THE RENEWAL-REWARD PROCESS

A. ABDULLAYEVA

Received: date / Revised: date / Accepted: date

Abstract. *In this manuscript renewal-reward process is investigated and an exact formula for the third moment of this process is obtained. Moreover, as main result, an asymptotic expansions as for the third moment is derived.*

Keywords: renewal-reward process, mathematical expectation, high-order moments, reward function

Mathematics Subject Classification (2020): 60H30, 60G50, 60K05

1. Introduction

The renewal-reward process belongs to the class of the stochastic processes of discrete interference of chance. There are many important results in the scientific literature related to this process (see, for example, [1]-[5], [7], [8], [10]). Renewal-reward processes are applied in insurance and reliability theories. In the present study, the asymptotic expansion for the third-order initial moment of the renewal-reward process is given.

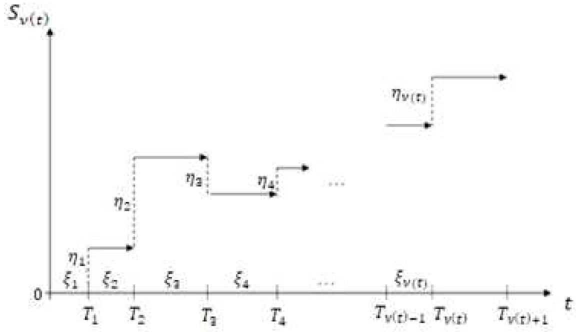
Let random vectors (ξ_n, η_n) , $n \geq 1$ be independent and identically distributed. In the general case, the random variable η_n is assumed to depend on the random variable ξ_n. Let's denote the distribution function of ξ_n by $F : F(x) = P\{\xi_n \leq x\}$.

Let's introduce the following sum:

$$S_{\nu(t)} = \sum_{n=1}^{\nu(t)} \eta_n, \tag{1}$$

where $\nu(t) = \max\{n : T_n \leq t\}$, $t > 0$ is the renewal process (see [6]) and $T_n = \sum_{i=1}^{n} \xi_i$, $n = 1, 2, \ldots$. The process $S_{\nu(t)}$, $t \geq 0$ is called as renewal-reward process and represents the sum of the rewards obtained up to time t. It is easy to see that, the renewal-reward process is a generalization of the renewal process. So, in the special case if $\eta_n \equiv 1$, $n \geq 1$ we obtain that $S_{\nu(t)} \equiv \nu(t)$.

Afaq Abdullayeva
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
E-mail: afaq.abdullayeva21@gmail.com



Fiq. 1. One of realizations of the renewal-reward process

2. Main Results

Denote the third order moment of the renewal-reward process (1) by $D_3(t)$ and the mathematical expectation of the renewal-reward process by $M_n(t)$ when the rewards are given by η_k^n $(k \geq 1, n \geq 1)$:

$$D_3(t) = E\left(S_{\nu(t)}\right)^3 = E\left(\sum_{k=1}^{\nu(t)} \eta_k\right)^3, \quad M_n(t) = E\left(\sum_{k=1}^{\nu(t)} \eta_k^n\right).$$

It is clear that, $D_1(t) = M_1(t) = D(t)$, where $D(t)$ is the mathematical expectation of the process (1).

Our main purpose is to obtain an asymptotic expansion for $D_3(t)$ as $t \to \infty$. For this, we will obtain exact formula for $D_3(t)$ in terms of $M_1(t)$, $M_2(t)$ and $M_3(t)$. Then using asymptotic expansions for $M_n(t)$, $n = 1, 2, 3$ will take us to our main purpose.

2. 1. Exact formula for $D_3(t)$ in terms of $M_n(t)$, $n = 1, 2, 3$

Let $F^{*(k)}$ be the k-th convolution of F, and let

$$f_k(t) = \frac{dF^{*(k)}(t)}{dH(t)},$$

where $H(t) = E(\nu(t))$ is a renewal function. Since

$$H(t) = \sum_{k=1}^{\infty} F^{*(k)}(t),$$

it follows that $H(t) = 0$ implies that all $F^{*(k)}(t) = 0$; thus $F^{*(k)}(t) \ll H(t)$, and $f_k(t)$ is well defined.

Denote

$$M_1 * M_2(t) = \int_0^t M_1(t-x)dM_2(x),$$

$$M_1^{*(3)}(t) = \int_0^t M_1^{*(2)}(t-x)dM_1(x).$$

To obtain exact formula for $D_3(t)$ in terms of $M_1(t)$, $M_2(t)$ and $M_3(t)$, we first need to prove the following lemma.

Lemma 1. *Let random vectors* (ξ_n, η_n) , $n \geq 1$ *be independent and identically distributed. In the general case, the random variable* η_n *is assumed to depend on the random variable* ξ_n. *Then*

$$E\left(\sum_{\substack{j\neq k\\ j,k\leq \nu(t)}} \eta_j^2\eta_k\right) = 2M_1 * M_2(t),$$

$$E\left(\sum_{i<j<k\leq\nu(t)} \eta_i\eta_j\eta_k\right) = M_1^{*(3)}(t).$$

Proof. Since

$$M_1(t) = E\left(\sum_{k=1}^{\nu(t)}\eta_k\right) = E\left(\sum_{k=1}^{\infty}\eta_k I\left\{T_k\leq t\right\}\right) =$$

$$= \sum_{k=1}^{\infty}\int_0^t E\left(\eta_k|T_k=s\right)dF^{*(k)}(s) = \int_0^t\left(\sum_{k=1}^{\infty}E\left(\eta_k|T_k=s\right)f_k(s)\right)dH(s)$$

and

$$M_2(t) = E\left(\sum_{k=1}^{\nu(t)}\eta_k^2\right) = E\left(\sum_{k=1}^{\infty}\eta_k^2 I\left\{T_k\leq t\right\}\right) =$$

$$= \sum_{k=1}^{\infty}\int_0^t E\left(\eta_k^2|T_k=s\right)dF^{*(k)}(s) = \int_0^t\left(\sum_{k=1}^{\infty}E\left(\eta_k^2|T_k=s\right)f_k(s)\right)dH(s).$$

Therefore

$$\frac{dM_1(s)}{dH(s)} = \sum_{k=1}^{\infty}E\left(\eta_k|T_k=s\right)f_k(s),$$

$$\frac{dM_2(s)}{dH(s)} = \sum_{k=1}^{\infty}E\left(\eta_k^2|T_k=s\right)f_k(s).$$

Next

$$E\left(\sum_{j<k\leq\nu(t)}\eta_j^2\eta_k\right) = E\left(\sum_{j<k}\eta_j^2\eta_k I\left\{T_k\leq t\right\}\right) =$$

$$= \sum_{j<k} \int_{w_1=0}^{t} \int_{w_2=w_1}^{t} E\left(\eta_j^2 \middle| T_j = w_1\right) E\left(\eta_{k-j} | T_{k-j} = w_2 - w_1\right) dF^{*(j)}\left(w_1\right) \times$$

$$\times dF^{*(k-j)}\left(w_2 - w_1\right) =$$

$$= \int_{w_1=0}^{t} \int_{w_2=w_1}^{t} \Big(\sum_j E\left(\eta_j^2 \middle| T_j = w_1\right) f_j\left(w_1\right) \times$$

$$\times \sum_k E\left(\eta_k | T_k = w_2 - w_1\right) f_k\left(w_2 - w_1\right) \times$$

$$\times dH\left(w_2 - w_1\right) dH\left(w_1\right) \Big) = \int_0^t M_2\left(t - w_1\right) \left(\frac{dM_1\left(w_1\right)}{dH\left(w_1\right)} \right) dH\left(w_1\right) =$$

$$= \int_0^t M_2\left(t - w_2\right) dM_1\left(w_1\right) = M_2 * M_1(t).$$

Similarly,

$$E\left(\sum_{k<j\le\nu(t)} \eta_j^2 \eta_k \right) = \int_0^t M_1\left(t - w_2\right) dM_2\left(w_1\right) = M_1 * M_2(t).$$

So,

$$E\left(\sum_{\substack{j\neq k \\ j,k\le\nu(t)}} \eta_j^2 \eta_k \right) = E\left(\sum_{j<k\le\nu(t)} \eta_j^2 \eta_k \right) + E\left(\sum_{k<j\le\nu(t)} \eta_j^2 \eta_k \right) = 2M_1 * M_2(t).$$

Next

$$E\left(\sum_{i<j<k\le\nu(t)} \eta_i \eta_j \eta_k \right) = E\left(\sum_{i<j<k} \eta_i \eta_j \eta_k I\left\{T_k \le t\right\} \right) =$$

$$= \sum_{i<j<k} \int_{w_1=0}^{t} \int_{w_2=w_1}^{t} \int_{w_3=w_2}^{t} \Big(E\left(\eta_i | T_i = w_1\right) E\left(\eta_{j-i} | T_{j-i} = w_2 - w_1\right) \times$$

$$\times E\left(\eta_{k-j} | T_{k-j} = w_3 - w_2\right) dF^{*(i)}\left(w_1\right) dF^{*(j-i)}\left(w_2 - w_1\right) dF^{*(k-j)}\left(w_3 - w_2\right) \Big) =$$

$$= \int_{w_1=0}^{t} \int_{w_2=w_1}^{t} \int_{w_3=w_2}^{t} \Big(\sum_i E\left(\eta_i | T_i = w_1\right) f_i\left(w_1\right) \times$$

$$\times \sum_j E\left(\eta_j | T_j = w_2 - w_1\right) f_j\left(w_2 - w_1\right) \times$$

$$\times \sum_k E\left(\eta_k | T_k = w_3 - w_2\right) f_k\left(w_3 - w_2\right) dH\left(w_3 - w_2\right) dH\left(w_2 - w_1\right) dH\left(w_1\right) =$$

$$= \int_{w_1=0}^{t} \int_{w_2=w_1}^{t} M_1 (t - w_2) \left(\frac{dM_1 (w_2 - w_1)}{dH (w_2 - w_1)} \right) \left(\frac{dM_1 (w_1)}{dH (w_1)} \right) dH (w_2 - w_1) \, dH (w_1) =$$

$$= \int_{w_1=0}^{t} \int_{w_2=w_1}^{t} M_1 (t - w_2) \, dM_1 (w_2 - w_1) \, dM_1 (w_1) = M_1 * M_1 * M_1 = M_1^{*(3)}(t).$$

This completes the proof of Lemma 1. ◀

Lemma 2. *Let random vectors* (ξ_n, η_n) , $n \geq 1$ *be independent and identically distributed. In the general case, the random variable* η_n *is assumed to depend on the random variable* ξ_n. *Then*

$$D_3(t) = 6M_1^{*(3)}(t) + 6M_1 * M_2(t) + M_3(t),$$

where

$$M_1 * M_2(t) = \int_0^t M_1(t - x) dM_2(x),$$

$$M_1^{*(3)}(t) = \int_0^t M_1^{*(2)}(t - x) dM_1(x).$$

Proof. Since

$$D_3(t) = E \left(\sum_{k=1}^{\nu(t)} \eta_k \right)^3 = E \left(\sum_{k=1}^{\nu(t)} \eta_k^3 \right) +$$

$$+3E \left(\sum_{\substack{j \neq k \\ j, k \leq \nu(t)}} \eta_j^2 \eta_k \right) + 6E \left(\sum_{i<j<k\leq\nu(t)} \eta_i \eta_j \eta_k \right), \qquad (2)$$

then using Lemma 1 yields us to the proof of this Lemma. ◀

2. 2. Asymptotic expansion for $D_3(t)$

Define

$$\mu_k = E\xi_1^k, \quad k \geq 0,$$

$$R(t) = H(t) - \frac{1}{\mu_1} t - \frac{\mu_2}{2\mu_1^2} + 1.$$

Set

$$r_k = \int_0^\infty t^k R(t) dt, \quad k \geq 0$$

whenever $\int_0^\infty t^k \left| R(t) \right| dt < \infty$.

Definition. *A distribution function* F *is said to belong to the class* ϑ *if some convolution of* F *has an absolutely continuous component.*

To obtain an asymptotic expansion for $D_3(t)$ we need to prove the following lemmas.

Lemma 3. *If $F \in \vartheta$ and $\mu_4 < \infty$, then*

$$r_1 = -\frac{\mu_2^3}{8\mu_1^4} + \frac{\mu_2\mu_3}{6\mu_1^3} - \frac{\mu_4}{24\mu_1^2}.$$

Proof. We will use similar method by Brown and Solomon [5]. From the conditions it is implied that [11, p. 2]

(i) $\lim\limits_{t\to\infty} t^2 R(t) = 0$;

(ii) $\int\limits_0^\infty t\,|R(t)|\,dt < \infty$;

(iii) $tR(t)$ is of bounded variation on $[\ 0, \infty)$.

Brown and Solomon [5] obtained the expression for r_0:

$$r_0 = \frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}.$$

For r_1 consider the functions

$$g_1(t) = t\int_{x=0}^{t} xR(t-x)dF(x),$$

$$g_2(t) = \int_{x=0}^{t} x^2R(t-x)dF(x).$$

By (ii) it can easily be proved that g_1 and g_2 are integrable and

$$\int_0^\infty g_1(t)dt = \mu_1 r_1 + \mu_2 r_0,$$

$$\int_0^\infty g_2(t)dt = \mu_2 r_0.$$

Also, using the same method in Brown and Solomon [5] it can be shown that g_1 and g_2 are of bounded variation on $[\ 0, \infty)$. Thus, g_1 and g_2 are directly Riemann integrable.

Now

$$H(t) = F(t) + \int_0^t H(t-x)dF(x).$$

Subtract $\frac{1}{\mu_1}t + \frac{\mu_2}{2\mu_1^2} - 1$ from both sides and then multiply by t^2 to obtain

$$t^2R(t) = \int_0^t (t-x)^2R(t-x)dF(x) + Z(t),$$

where

$$Z(t) = 2g_1 - g_2 + \frac{1}{\mu_1}t^2\int_t^\infty (1-F(t))\,dt - \frac{\mu_2}{2\mu_1^2}t^2\,(1-F(t))\,.$$

By the key renewal theorem

$$\lim_{t\to\infty} t^2 R(t) = \frac{1}{\mu_1}\int_0^\infty Z(t)dt = \frac{1}{\mu_1}\left(2\mu_1 r_1 + \mu_2 r_0 + \frac{\mu_4}{12\mu_1} - \frac{\mu_2\mu_3}{6\mu_1^2}\right).$$

But, by (i), $\lim\limits_{t\to\infty} t^2 R(t) = 0$, thus

$$r_1 = -\frac{\mu_2}{2\mu_1} r_0 - \frac{\mu_4}{24\mu_1^2} + \frac{\mu_2\mu_3}{12\mu_1^3} = -\frac{\mu_2^3}{8\mu_1^4} + \frac{\mu_2\mu_3}{6\mu_1^3} - \frac{\mu_4}{24\mu_1^2}.$$

This completes the proof of Lemma 3. ◄

Define

$$\lambda_s = E\eta_1^s = \int_0^\infty E\left(\eta_1^s | \xi_1 = t\right) dF(t), \quad n_{k,s} = E\left(\xi_1^k \eta_1^s\right) = \int_0^\infty x^k E\left(\eta_1^s | \xi_1 = x\right) dF(x).$$

It is clear that $n_{k,0} = \mu_k$, $n_{0,s} = \lambda_s$.
Define

$$L(t) = D(t) - at - b,$$

where $a = \frac{\lambda_1}{\mu_1}$, $b = \frac{\lambda_1\mu_2}{2\mu_1^2} - \frac{n_{1,1}}{\mu_1}$.
Set

$$l_s = \int_0^\infty t^s L(t)dt, \quad s \geq 0,$$

whenever $\int_0^\infty t^s |L(t)|\, dt < \infty$.

Lemma 4. *$F \in \vartheta$ and μ_4, λ_1, $n_{3,1}$ exist. Then*

$$l_1 = \lambda_1 r_1 + n_{1,1} r_0 + \frac{n_{3,1}}{6\mu_1} - \frac{\mu_2 n_{2,1}}{4\mu_1^2}. \tag{3}$$

Proof. Note that since λ_1 and $n_{3,1}$ exist, then $n_{1,1}$ and $n_{2,1}$ also exist. Also,

$$S_{\nu(t)} = \sum_{i=1}^{\nu(t)} \eta_i = \sum_{i=1}^{\nu(t)+1} \eta_i - \eta_{\nu(t)+1}.$$

Since $\nu(t)+1$ is a stopping time [9]

$$D(t) = \lambda_1 \left(H(t) + 1\right) - E\left(\eta_{\nu(t)+1}\right).$$

Subtracting $at + b$ from both sides and multiplying by t we obtain

$$tL(t) = \lambda_1 t R(t) + t\left(\frac{n_{1,1}}{\mu_1} - E\left(\eta_{\nu(t)+1}\right)\right).$$

Thus, if $t\left(\frac{n_{1,1}}{\mu_1} - E\left(\eta_{\nu(t)+1}\right)\right)$ is integrable, then

$$l_1 = \lambda_1 r_1 + \int_0^\infty t\left(\frac{n_{1,1}}{\mu_1} - E\left(\eta_{\nu(t)+1}\right)\right) dt.$$

But

$$\frac{n_{1,1}}{\mu_1} = \frac{1}{\mu_1}\int_0^\infty xE\left(\eta_1|\xi_1 = x\right)dF(x) \tag{4}$$

and [10, p. 134]. So, it is not difficult to see that,

$$E\left(\eta_{\nu(t)+1}\right) = \int_t^\infty E\left(\eta_1|\xi_1 = x\right)dF(x) + \int_0^\infty E\left(\eta_1|\xi_1 = x\right)\left(H(t) - H(t-x)\right)dF(x).$$

It follows from (3) and (4) that

$$\frac{n_{1,1}}{\mu_1} - E\left(\eta_{\nu(t)+1}\right) = \int_0^\infty E\left(\eta_1|\xi_1 = x\right)\left(R(t-x) - R(t)\right)dF(x) -$$

$$- \int_t^\infty E\left(\eta_1|\xi_1 = x\right)dF(x).$$

Thus

$$\int_0^\infty t\left|\frac{n_{1,1}}{\mu_1} - E\left(\eta_{\nu(t)+1}\right)\right|dt \le 2E\left|\eta_1\right|\int_0^\infty t\left|R(t)\right|dt + E\left(\xi_1\left|\eta_1\right|\right)\int_0^\infty \left|R(t)\right|dt +$$

$$+ \frac{5}{6\mu_1}E\left(\xi_1^3\left|\eta_1\right|\right) + \frac{3}{2}\left(\frac{\mu_2}{2\mu_1^2} + 1\right)E\left(\xi_1^2\left|\eta_1\right|\right) + \frac{1}{2}E\left(\xi_1^2\left|\eta_1\right|\right) < \infty.$$

Thus $t\left(\frac{n_{1,1}}{\mu_1} - E\left(\eta_{\nu(t)+1}\right)\right)$ is integrable, and

$$\int_0^\infty t\left(\frac{n_{1,1}}{\mu_1} - E\left(\eta_{\nu(t)+1}\right)\right)dt = n_{1,1}r_0 - \frac{\mu_2}{4\mu_1^2}n_{2,1} + \frac{n_{3,1}}{6\mu_1}.$$

This completes the proof of Lemma 4. ◄

Denote

$$L_k(t) = M_k(t) - a_k t - b_k,$$

where $a_k = \frac{\lambda_k}{\mu_1}$, $b_k = \frac{\lambda_k\mu_2}{2\mu_1^2} - \frac{n_{1,k}}{\mu_1}$.

Set

$$l_{k,s} = \int_0^\infty t^s L_k(t)dt, \quad s = 0, 1,$$

whenever $\int_0^\infty t^s\left|L_k(t)\right|dt < \infty$.

Lemma 5. *Let $F(x)$ be a strongly non-lattice distribution function, $F \in \vartheta$ and λ_n, μ_4, $n_{3,n}$ exist, where $n = \max(i,j)$, $i, j \ge 1$. Then*

$$M_i * M_j(t) = a_{i*j}t^2 + b_{i*j}t + c_{i*j} + L_{i*j}(t),$$

where $a_{i*j} = \frac{1}{2}a_i a_j$, $b_{i*j} = a_j b_i + a_i b_j$, $c_{i*j} = b_i b_j + a_j l_{i,0} + a_i l_{j,0}$,

$$L_{i*j}(t) = o\left(t^{-1}\right), \quad t \to \infty.$$

Proof. According to the Theorem 2.1 from [1], we can write

$$M_k(t) = a_k t + b_k + L_k(t), \quad k = i, j,$$

where $a_k = \frac{\lambda_k}{\mu_1}, \quad b_k = \frac{\lambda_k \mu_2}{2\mu_1^2} - \frac{n_{1,k}}{\mu_1}, \quad L_k(t) = o\left(t^{-2}\right), \quad t \to \infty.$

Also, from Lemma 4,

$$\int_0^\infty t^s \left|L_k(t)\right| dt < \infty, \quad s = 0, 1; \quad k = i, j.$$

Then

$$M_i * M_j(t) = \int_0^t M_i(t-x) dM_j(x) = \frac{1}{2} a_i a_j t^2 + \left(a_j b_i + a_i b_j\right) t + b_i b_j +$$

$$+ a_j l_{i,0} - a_j \int_t^\infty L_i(x) dx + a_i l_{j,0} - a_i \int_t^\infty L_j(x) dx + b_i L_j(t) +$$

$$+ \int_0^t L_i(t-x) dL_j(x) = a_{i*j} t^2 + b_{i*j} t + c_{i*j} + L_{i*j}(t),$$

where

$$L_{i*j}(t) = -a_j \int_t^\infty L_i(x) dx - a_i \int_t^\infty L_j(x) dx + b_i L_j(t) + \int_0^t L_i(t-x) dL_j(x),$$

$$a_{i*j} = \frac{1}{2} a_i a_j, \quad b_{i*j} = a_j b_i + a_i b_j, \quad c_{i*j} = b_i b_j + a_j l_{i,0} + a_i l_{j,0}.$$

It is not difficult to see that

$$\left| t \int_t^\infty L_k(x) dx \right| \le \int_t^\infty t \left|L_k(x)\right| dx \to 0, \quad t \to \infty.$$

Therefore,

$$\int_t^\infty L_k(x) dx = o\left(t^{-1}\right), \quad t \to \infty, \quad k = i, j.$$

For the third term of $L_{i*j}(t)$, we will use Laplace transform. Denote

$$A(t) = t \int_0^t L_i(t-x) dL_j(x).$$

Denote Laplace transform of $A(t)$ by $\widehat{A}(\alpha)$:

$$\widehat{A}(\alpha) = \int_0^\infty e^{-\alpha t} A(t) dt = \int_{x=0}^\infty e^{-\alpha x} dL_j(x) \int_{t=0}^\infty t e^{-\alpha t} L_i(t) dt +$$

$$+ \int_{x=0}^\infty x e^{-\alpha x} dL_j(x) \int_{t=0}^\infty e^{-\alpha t} L_i(t) dt.$$

Since $\lim_{\alpha\to 0} \alpha\widehat{A}(\alpha) = 0$, thus by the Tauberian theorem $\lim_{t\to\infty} A(t) = 0$. Therefore,

$$\int_0^t L_i(t-x)dL_j(x) = o\left(t^{-1}\right), \quad t\to\infty.$$

Consequently,

$$L_{i*j}(t) = o\left(t^{-1}\right), \quad t\to\infty.$$

This completes the proof of Lemma 5. ◀

Corollary 1. *Let $F(x)$ be a strongly non-lattice distribution function, $F\in\vartheta$ and λ_2, μ_4, $n_{3,2}$ exist. Then as $t\to\infty$*

$$M_1 * M_2(t) = a_{1*2}t^2 + b_{1*2}t + c_{1*2} + L_{1*2}(t),$$

where $a_{1*2} = \frac{1}{2}a_1a_2$, $b_{1*2} = a_2b_1 + a_1b_2$, $c_{1*2} = b_1b_2 + a_2l_{1,0} + a_1l_{2,0}$, $L_{1*2}(t) = o\left(t^{-1}\right)$ as $t\to\infty$.

Corollary 2. *Let $F(x)$ be a strongly non-lattice distribution function, $F\in\vartheta$ and λ_k, μ_4, $n_{3,k}$ exist. Then as $t\to\infty$*

$$M_k^{*(2)}(t) = a_k^{*(2)}t^2 + b_k^{*(2)}t + c_k^{*(2)} + L_k^{*(2)}(t),$$

where $a_k^{*(2)} = \frac{1}{2}a_k^2$, $b_k^{*(2)} = 2a_kb_k$, $c_k^{*(2)} = b_k^2 + 2a_kl_{k,0}$, $L_k^{*(2)}(t) = o\left(t^{-1}\right)$, as $t\to\infty$.

Lemma 6. *Let $F(x)$ be a strongly non-lattice distribution function, $F\in\vartheta$ and λ_1, μ_4, $n_{3,1}$ exist. Then as $t\to\infty$*

$$M_1^{*(3)}(t) = a_1^{*(3)}t^3 + b_1^{*(3)}t^2 + c_1^{*(3)}t + d_1^{*(3)} + L_1^{*(3)}(t),$$

where $a_1^{*(3)} = \frac{1}{3}a_1a_1^{*(2)}$, $b_1^{*(3)} = \frac{1}{2}a_1b_1^{*(2)} + a_1^{*(2)}b_1$, $c_1^{*(3)} = a_1c_1^{*(2)} + b_1b_1^{*(2)} + 2a_1^{*(2)}l_{1,0}$, $d_1^{*(3)} = a_1l_1^{*(2)} + b_1c_1^{*(2)} + b_1^{*(2)}l_{1,0} - 2a_1^{*(2)}l_{1,1}$ and $L_1^{*(3)}(t) = o(1)$, $t\to\infty$.

Proof. Since

$$M_1^{*(3)}(t) = \int_0^t M_1^{*(2)}(t-x)dM_1(x),$$

by using Theorem 2.1 from [1] and Corollary 2, we can write:

$$M_1^{*(3)}(t) = \frac{1}{3}a_1a_1^{*(2)}t^3 + \left(\frac{1}{2}a_1b_1^{*(2)} + a_1^{*(2)}b_1\right)t^2 + \left(a_1c_1^{*(2)} + b_1b_1^{*(2)} + 2a_1^{*(2)}l_{1,0}\right)t +$$

$$+\left(a_1l_1^{*(2)} + b_1c_1^{*(2)} + b_1^{*(2)}l_{1,0} - 2a_1^{*(2)}l_{1,1}\right) - a_1\int_t^\infty L_1^{*(2)}(x)dx -$$

$$-\left(2a_1^{*(2)}t + b_1^{*(2)}\right)\left(tL_1(t) + \int_t^\infty L_1(x)dx\right) + a_1^{*(2)}\left(t^2L_1(t) + 2\int_t^\infty xL_1(x)dx\right) +$$

$$+\int_0^t L_1^{*(2)}(t-x)dL_1(x) = a_1^{*(3)}t^3 + b_1^{*(3)}t^2 + c_1^{*(3)}t + d_1^{*(3)} + L_1^{*(3)}(t),$$

where $a_1^{*(3)} = \frac{1}{3} a_1 a_1^{*(2)}$, $b_1^{*(3)} = \frac{1}{2} a_1 b_1^{*(2)} + a_1^{*(2)} b_1$, $c_1^{*(3)} = a_1 c_1^{*(2)} + b_1 b_1^{*(2)} + 2a_1^{*(2)} l_{1,0}$, $d_1^{*(3)} = a_1 l_1^{*(2)} + b_1 c_1^{*(2)} + b_1^{*(2)} l_{1,0} - 2a_1^{*(2)} l_{1,1}$ and

$$L_1^{*(3)}(t) = -a_1^{*(2)} t^2 L_1(t) - b_1^{*(2)} t L_1(t) - a_1 \int_t^\infty L_1^{*(2)}(x) dx -$$

$$- \left(2a_1^{*(2)} t + b_1^{*(2)}\right) \int_t^\infty L_1(x) dx + 2a_1^{*(2)} \int_t^\infty x L_1(x) dx + \int_0^t L_1^{*(2)}(t-x) dL_1(x).$$

Since $L_1(t) = o\left(t^{-2}\right)$ as $t \to \infty$, then

$$t L_1(t) = o\left(t^{-1}\right), \quad t^2 L_1(t) = o(1), \quad t \to \infty.$$

It is not difficult to prove that $L_1^{*(2)}(t)$ is integrable and

$$l_1^{*(2)} = \int_0^\infty L_1^{*(2)}(t) dt = -2a_1 l_{1,1} + 2b_1 l_{1,0}.$$

By taking into account this and Lemma 4, we can write

$$\int_t^\infty L_1^{*(2)}(x) dx = o(1), \quad \int_t^\infty x L_1(x) dx = o(1), \quad \int_t^\infty L_1(x) dx = o\left(t^{-1}\right), \quad t \to \infty.$$

Denote the Laplace transform of the last term of $L_1^{*(3)}(t)$ by $\widehat{L}_3(\alpha)$:

$$\widehat{L}_3(\alpha) = \int_{t=0}^\infty e^{-\alpha t} \int_{x=0}^t L_1^{*(2)}(t-x) dL_1(x) dt = \widehat{L}_1(\alpha) \cdot \widehat{L}_2(\alpha),$$

where $\widehat{L}_1(\alpha)$ and $\widehat{L}_2(\alpha)$ are the Laplace transforms of $L_1(t)$ and $L_1^{*(2)}(t)$, respectively:

$$\widehat{L}_1(\alpha) = \int_{t=0}^\infty e^{-\alpha t} \int_{x=0}^t L_1(x) dx dt, \quad \widehat{L}_2(\alpha) = \int_{t=0}^\infty e^{-\alpha t} \int_{x=0}^t L_1^{*(2)}(x) dx dt.$$

Since $\lim\limits_{\alpha \to 0} \alpha \widehat{L}(\alpha) = 0$, so by the Tauberian theorem $\lim\limits_{t \to \infty} \int_0^t L_1^{*(2)}(t-x) dL_1(x) = 0$, thus

$$L_1^{*(3)}(t) = \int_0^t L_1^{*(2)}(t-x) dL_1(x) = o(1), \quad t \to \infty.$$

This completes the proof of Lemma 6. ◄

Theorem. *Let $F(x)$ be a strongly non-lattice distribution function, $F \in \vartheta$ and λ_3, μ_4, $n_{3,3}$ exist. Then as $t \to \infty$.*

$$D_3(t) = D_{3,3} t^3 + D_{3,2} t^2 + D_{3,1} t + D_{3,0} + o(1),$$

where

$$D_{3,3} = 6a_1^{*(3)} = \frac{\lambda_1^3}{\mu_1^3},$$

$$D_{3,2}=6b_1^{*(3)}+6a_{1*2}=\frac{9\lambda_1^2}{\mu_1^2}\left(\frac{\lambda_1\mu_2}{2\mu_1^2}-\frac{n_{1,1}}{\mu_1}\right)+\frac{3\lambda_1\lambda_2}{\mu_1^2}=\frac{9\lambda_1^3\mu_2}{2\mu_1^4}-\frac{9\lambda_1^2 n_{1,1}}{\mu_1^3}+\frac{3\lambda_1\lambda_2}{\mu_1^2},$$

$$D_{3,1}=6c_1^{*(3)}+6b_{1*2}+a_3=$$

$$=\frac{9\lambda_1^3\mu_2^2}{\mu_1^5}-\frac{54\lambda_1^2\mu_2 n_{1,1}+3\lambda_1^3\mu_3}{\mu_1^4}+\frac{18\lambda_1 n_{1,1}^2+9\lambda_1^2 n_{2,1}+6\lambda_1\lambda_2\mu_2}{\mu_1^3}-\frac{6\lambda_2 n_{1,1}+6\lambda_1 n_{1,2}}{\mu_1^2}+\frac{\lambda_3}{\mu_1},$$

$$D_{3,0}=6d_1^{*(3)}+6c_{1*2}+b_3=$$

$$=\frac{6\lambda_1^3\mu_2^3}{\mu_1^6}-\frac{27\lambda_1^2\mu_2^2 n_{1,1}+3\lambda_1^3\mu_2+3\lambda_1^3\mu_2\mu_3}{\mu_1^5}+$$

$$+\frac{54\lambda_1\mu_2 n_{1,1}^2+24\lambda_1^2 n_{1,1}+54\lambda_1^2\mu_2 n_{2,1}+3\lambda_1^3\mu_4+12\lambda_1^2\mu_3 n_{1,1}-18\lambda_1\lambda_2\mu_2^2}{4\mu_1^4}-$$

$$-\frac{6n_{1,1}^3+18\lambda_1 n_{1,1}n_{2,1}+6\lambda_2\mu_2 n_{1,1}+6\lambda_1\mu_2 n_{1,2}+2\lambda_1\lambda_2\mu_3}{\mu_1^3}+$$

$$+\frac{12n_{1,1}n_{1,2}+6\lambda_2 n_{2,1}+6\lambda_1 n_{2,2}+\lambda_3\mu_2}{2\mu_1^2}-\frac{3\lambda_1^2 n_{3,1}+n_{1,3}}{\mu_1}.$$

Proof. From Theorem we can write

$$D_3(t)=6M_1^{*(3)}(t)+6M_2*M_1(t)+M_3(t).$$

From Corollary 1 we can write

$$M_1*M_2(t)=a_{1*2}t^2+b_{1*2}t+c_{1*2}+L_{1*2}(t),\quad L_{1*2}(t)=o\left(t^{-1}\right),\quad t\to\infty.$$

From Lemma 6 we can write

$$M_1^{*(3)}(t)=a_1^{*(3)}t^3+b_1^{*(3)}t^2+c_1^{*(3)}t+d_1^{*(3)}+L_1^{*(3)}(t),\quad L_1^{*(3)}(t)=o(1),\quad t\to\infty.$$

From the Theorem 2.1 of [1], we can write

$$M_3(t)=a_3t+b_3+L_3(t),\quad L_3(t)=o\left(t^{-2}\right),\quad t\to\infty.$$

By combining this results, we can write

$$D_3(t)=D_{3,3}t^3+D_{3,2}t^2+D_{3,1}t+D_{3,0}+L_{D_3}(t),$$

where $D_{3,3}=6a_1^{*(3)}$, $D_{3,2}=6b_1^{*(3)}+6a_{1*2}$, $D_{3,1}=6c_1^{*(3)}+6b_{1*2}+a_3$, $D_{3,0}=6d_1^{*(3)}+6c_{1*2}+b_3$ and

$$L_{D_3}(t)=6L_1^{*(3)}(t)+6L_{1*2}(t)+L_3(t)=o(1)+o\left(t^{-1}\right)+o\left(t^{-2}\right)=o(1),\quad t\to\infty.$$

This completes the proof of Theorem. ◀

Acknowledgement The author expresses her thanks to Professor R.T.Aliyev, Baku State University, and Ph.D. in Mathematics V.A.Bayramov, Space Agency of the Republic of Azerbaijan (Azercosmos), for their support and valuable advices.

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