Baku Mathematical Journal
2024, Vol. 3, No 2, PP. 220-227
https://doi.org/10.32010/j.bmj.2024.21

THE BOUNDEDNESS OF THE MAXIMAL OPERATOR IN LOCAL "COMPLEMENTARY" GENERALIZED MORREY SPACES, ASSOCIATED WITH THE LAPLACE-BESSEL DIFFERENTIAL OPERATOR

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Received: 15.05.2024 / Revised: 19.07.2024 / Accepted: 30.07.2024

In memory of M. G. Gasymov on his 85th birthday

Abstract. *We consider the generalized shift operator, associated with the Laplace-Bessel differential operator. The maximal operator, associated with the generalized shift operator are investigated. We prove that the B-maximal operator is bounded from the local "complementary" generalized B-Morrey space* ${}^{\complement}\mathcal{M}_{\{0\}}^{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ *to* ${}^{\complement}\mathcal{M}_{\{0\}}^{p,\omega_2,\gamma}(\mathbb{R}^n_{k,+})$ *where* $1 < p < \infty$.

Keywords: B-maximal operator, B-Riesz potential, generalized B-Morrey space, local "complementary" generalized B-Morrey space

Mathematics Subject Classification (2020): 42B20, 42B25, 42B35

1. Introduction

In 1938, C. Morrey [18] considered the integral growth condition on derivatives over balls, in order to study the existence and regularity for partial differential equations. A family of functions with the integral growth condition is then called a Morrey space after his name. In the study of local properties of solutions of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ play an important

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role (see [14]). Morrey spaces are defined by the norm

$$\|f\|_{\mathcal{M}^{p,\lambda}} := \sup_{x,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $0 \leq \lambda < n$, $1 \leq p < \infty$ and $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$.

Until recently, a rapid growth has been seen in the study of Morrey-type spaces because of its applications in major fields of engineering and sciences. Function spaces with non-standard growth has seen a major focus in recent times because of its wide range of applications in the area of image processing, the study of thermorheological fluids and modeling of electrorheological fluids. The theory of boundedness of classical operators of Real Analysis, such as maximal operator, fractional maximal operator, Riesz potential, singular integral operator etc, is by now well studied. These results can be applied fruitful in the theory of partial differential equations. It should be noted that in the theory of partial differential equations, in the last years, general Morrey-type spaces play an important role. In the nineties of the XX century an extensive study of general Morrey type spaces, characterized by a functional parameter, started. In particular, V.S. Guliev in his doctoral thesis (1994) introduced local and complementary local Morrey-type spaces and studied the boundedness in these spaces of fractional integral operators and singular integral operators defined on homogeneous Lie groups. A number of results on boundedness of classical operators in general Morrey type spaces were obtained by However in all these results only sufficient conditions on the functional parameters, characterizing general Morrey-type spaces, ensuring boundedness, were obtained. At the beginning of the XXI century there were new deep developments in this research area. In particular, V.S. Guliev, jointly with V.I. Burenkov, has developed a new perspective trend in harmonic analysis, related to the study of classical operators in general spaces of Morrey type. The significance of the developed methods lies in the fact that they allow to obtain necessary and sufficient conditions for the boundedness of classes of singular type operators with the subsequent application to regularity theory for solutions to elliptic and parabolic partial differential equations. As a result, for a certain range of the numerical parameters necessary and sufficient conditions were obtained on the functional parameters ensuring boundedness of classical operators of Real Analysis (maximal operator, fractional maximal operator, Riesz potential, genuine singular integrals) from one general local Morrey-type space to another one. Results of such type are very important for the development of contemporary Real Analysis and its applications, first of all, to Partial Differential Equations.

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r.

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. The maximal operator M is defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)| dy,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$.

The operator M play an important role in real and harmonic analysis (see, for example [23] and [21]).

These spaces appeared to be quite useful in the study of the local behaviour of the solutions to elliptic partial differential equations, apriori estimates and other topics in the theory of partial differential equations.

D. R. Adams [1] studied the boundedness of the Riesz potential in Morrey spaces. F. Chiarenza and M. Frasca [4] studied the boundedness of the maximal operator in Morrey spaces.

If in place of the power function r^λ in the definition of $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ we consider any positive measurable weight function $\omega(r)$, then it becomes generalized Morrey space $\mathcal{M}^{p,\omega}(\mathbb{R}^n)$.

Definition 1. *Let $\omega(r)$ positive measurable weight function on $(0,\infty)$ and $1 \le p < \infty$. We denote by $\mathcal{M}^{p,\omega}(\mathbb{R}^n)$ the generalized Morrey spaces, the spaces of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{\mathcal{M}^{p,\omega}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{r^{-\frac{n}{p}}}{\omega(r)} \|f\|_{L_p(B(x,r))}.$$

T. Mizuhara [17], E. Nakai [20] and V. S. Guliev [6] obtained sufficient conditions on weights ω_1 and ω_2 ensuring the boundedness of T from $\mathcal{M}^{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}^{p,\omega_2}(\mathbb{R}^n)$. In [20] the following statement was proved, containing the result in [17].

In [6] local "complementary" generalized Morrey spaces ${}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\mathbb{R}^n)$ with constant p, the space of all functions $f \in L_p(\mathbb{R}^n \backslash B(x_0,r))$, $r>0$ by the norm

$$\|f\|_{{}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\mathbb{R}^n)} = \sup_{r>0} \frac{r^{\frac{n}{p'}}}{\omega(r)} \|f\|_{L_p(\mathbb{R}^n \backslash B(x_0,r))}$$

were introduced and studied.

We denote by [6] the local "complementary" Morrey spaces ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)$ with constant p, the space of all functions $f \in L_p(\mathbb{R}^n \backslash B(x_0,r))$, $r>0$ with the finite norm

$$\|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)} = \sup_{r>0} r^{\frac{\lambda}{p'}} \|f\|_{L_p(\mathbb{R}^n \backslash B(x_0,r))} < \infty, \quad x_0 \in \mathbb{R}^n,$$

where $1 \le p < \infty$ and $0 \le \lambda < n$. Note that ${}^{\complement}\mathcal{L}^{p,0}_{\{x_0\}}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$.

According to the definition of ${}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\mathbb{R}^n)$, we recover the space ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)$ under the choice $\omega(r) = r^{\frac{n-\lambda}{p'}}$:

$${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n) = {}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\mathbb{R}^n)\Big|_{\omega(r)=r^{\frac{n-\lambda}{p'}}}.$$

The maximal operator, potential and related topics associated with the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{k} \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad \gamma_1 > 0, \ldots, \gamma_k > 0$$

have been investigated by many researchers, see, for example [2], [3], [5], [8]-[13], [16], [19], [22], [24], [25] and others.

2. Preliminaries

Suppose that $\mathbb{R}^n$ is n-dimensional Euclidean space, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $|x|^2 = \sum_{i=1}^{n} x_i^2$, $x' = (x_1, \ldots, x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-k}$, $x = (x', x'') \in \mathbb{R}^n$, $n \geq 2$, $\mathbb{R}^n_{k,+} = \{x = (x', x'') \in \mathbb{R}^n;\ x_1 > 0, \ldots, x_k > 0\}$, $1 \leq k \leq n$, $E(x, r) = \{y \in \mathbb{R}^n_{k,+}\ ;\ |x - y| < r\}$, $E_r = E(0, r)$, $\gamma = (\gamma_1, \ldots, \gamma_k)$, $\gamma_1 > 0, \ldots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \ldots + \gamma_k$, $(x')^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$.

For measurable $E \subset \mathbb{R}^n_{k,+}$ suppose $|E|_\gamma = \int_E (x')^\gamma dx$, then $|E_r|_\gamma = \omega(n, k, \gamma) r^Q$, $Q = n + |\gamma|$, where

$$\omega(n, k, \gamma) = \int_{E_1} (x')^\gamma dx = \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^{k} \frac{\Gamma\left(\frac{\gamma_i + 1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}.$$

Denote by T^x the generalized shift operator (B–shift operator) acting according to the law

$$T^x f(y) = C_{\gamma,k} \int_0^\pi \cdots \int_0^\pi f\left((x', y')_\beta, x'' - y''\right)\ d\nu(\beta),$$

where $(x_i, y_i)_{\beta_i} = (x_i^2 - 2x_i y_i \cos\beta_i + y_i^2)^{\frac{1}{2}}$, $1 \leq i \leq k$, $(x', y')_\beta = ((x_1, y_1)_{\beta_1}, \ldots, (x_k, y_k)_{\beta_k})$, $d\nu\,(\beta) = \prod_{i=1}^{k} \sin^{\gamma_i - 1} \beta_i\ d\beta_1 \ldots d\beta_k$, $1 \leq k \leq n$ and

$$C_{\gamma,k} = \pi^{-\frac{k}{2}} \prod_{i=1}^{k} \frac{\Gamma\left(\frac{\gamma_i + 1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)} = \frac{2^k}{\pi^k} \omega(2k, k, \gamma).$$

We remark that the generalized shift operator T^x is closely connected with the Bessel differential operator B (for example, $n = k = 1$ see [15], $n > 1$, $k = 1$ see [13] and $n, k > 1$ see [16] for details).

Let $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ be the space of measurable functions on $\mathbb{R}^n_{k,+}$ with finite norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})} = \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx \right)^{1/p}, \quad 1 \leq p < \infty.$$

For $p = \infty$ the space $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ is defined by means of the usual modification

$$\|f\|_{L_{\infty,\gamma}} = \underset{x \in \mathbb{R}^n_{k,+}}{ess\ sup}\ |f(x)|.$$

The translation operator T^y generates the corresponding B-convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y) [T^x g(y)] (y')^\gamma dy,$$

for which the Young inequality

$$\|f \otimes g\|_{L_{r,\gamma}} \leq \|f\|_{L_{p,\gamma}} \|g\|_{L_{q,\gamma}}, \quad 1 \leq p, q \leq r \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1,$$

holds.

Definition 2. *[8] Let $1 \le p < \infty$, $0 \le \lambda \le Q$. We denote by $\mathcal{M}^{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ Morrey space ($\equiv$ B-Morrey space), associated with the Laplace-Bessel differential operator the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_{k,+}$, with the finite norm*

$$\|f\|_{\mathcal{M}^{p,\lambda,\gamma}} = \sup_{t>0,\, x\in\mathbb{R}^n_{k,+}} \left(t^{-\lambda} \int_{E_t} T^y[|f|\,]^p(x)(y')^\gamma dy \right)^{1/p}.$$

Consider the B-maximal operator

$$M_\gamma f(x) = \sup_{r>0} |E_r|_\gamma^{-1} \int_{E_r} T^y[|f|\,](x)(y')^\gamma dy.$$

Theorem 1. *[9] Let $1 < p < \infty$. Then the operator M_γ is bounded $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ spaces.*

Let $1 \le p < \infty$, ω positive measurable function. The norm in the spaces $\mathcal{M}^{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ defined by

$$\|f\|_{\mathcal{M}^{p,\omega,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \frac{t^{-\frac{Q}{p}}}{\omega(t)} \left(\int_{E_t} T^y[|f|\,]^p(x)(y')^\gamma dy \right)^{1/p},$$

the local "complementary" generalized B-Morrey space ${}^{\complement}\mathcal{M}^{p,\omega,\gamma}_{\{0\}}(\mathbb{R}^n_{k,+})$ is defined by the norm

$$\|f\|_{{}^{\complement}\mathcal{M}^{p,\omega,\gamma}_{\{0\}}(\mathbb{R}^n_{k,+})} = \sup_{t>0} \frac{t^{\frac{Q}{p'}}}{\omega(t)} \left(\int_{\mathbb{R}^n_{k,+}\setminus E(0,t)} T^y[|f|\,]^p(x)(y')^\gamma dy \right)^{1/p}.$$

If $\omega(t) \equiv r^{-\frac{Q}{p}}$ then $\mathcal{M}^{p,\omega,\gamma}(\mathbb{R}^n_{k,+}) \equiv L_{p,\gamma}(\mathbb{R}^n_{k,+})$, if $\omega(t) \equiv t^{\frac{\lambda-Q}{p}}$, $0 \le \lambda < Q$, then $\mathcal{M}^{p,\omega,\gamma}(\mathbb{R}^n_{k,+}) \equiv \mathcal{M}^{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

3. B-maximal Operator in the Spaces ${}^{\complement}\mathcal{M}^{p,\omega,\gamma}_{\{0\}}(\mathbb{R}^n_{k,+})$

Theorem 2. *Let $1 < p < \infty$ and $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,r)$ for every $r \in (0,\infty)$. If the integral*

$$\int_0^1 \|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,r)} \frac{dr}{r}$$

is convergent, then

$$\|M_\gamma f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,r)} \le C \int_0^{2t} \|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,s)} \frac{ds}{s} \tag{1}$$

for every $t \in (0,\infty)$, where C does not depend on f, t and x.

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{\mathbb{R}^n_{k,+}\setminus E(0,t)}(y), \quad f_2(y) = f(y)\chi_{E(0,t)}(y).$$

Frist estimation of $M_\gamma f_1$. This case is easier, being treated by means of Theorem 1. Obviously $f_1 \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ so that from Theorem 1 we get

$$\|M_\gamma f_1\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,t)} \le \|M_\gamma f_1\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})} \le$$

$$\le C\|f_1\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})} = C\|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,t)}. \tag{2}$$

Then in view of (2) we obtain

$$\|M_\gamma f_1\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,t)} \le C\int_0^t \|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,s)}\frac{ds}{s}. \tag{3}$$

Now estimation of $M_\gamma f_2$. To estimate $M_\gamma f_2(z)$, we observe that for $z \in \mathbb{R}^n_{k,+}\setminus E(0,t)$ we have

$$M_\gamma f_2(z) = \sup_{r>0} |E(0,r)|^{-1}\int_{E(0,r)} T^z|f_2(y)|(y')^\gamma dy \le$$

$$\le \sup_{r\ge t}\int_{E(0,t)\cap E(0,r)} |y|^{-n}\, T^z|f(y)|(y')^\gamma dy \le Ct^{-n}\int_{E(0,t)} T^z|f(y)|(y')^\gamma dy.$$

We use the following trick, in which the parameter $\beta > 0$ will be chosen later:

$$\int_{E(0,t)} T^z|f(y)|(y')^\gamma dy = \beta\int_{E(0,t)} |y|^{-\beta}T^z|f(y)|\left(\int_0^{|y|} s^{\beta-1}ds\right)(y')^\gamma dy =$$

$$= \beta\int_0^t s^{\beta-1}\left(\int_{\{y\in\mathbb{R}^n_{k,+}: s<|y|<t\}} |y|^{-\beta}\, T^z|f(y)|(y')^\gamma dy\right)ds.$$

Applying Hölder inequality we get

$$\int_{E(0,t)} T^z|f(y)|(y')^\gamma dy \le C\int_0^t \|T^z f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,s))}\, \|\chi_{E(0,t)}\|_{L_{p',\gamma}(\mathbb{R}^n_{k,+})}\frac{ds}{s}. \tag{4}$$

Then by (4)

$$M_\gamma f_2(z) \le C\,t^{-n}\,\|\chi_{E(0,t)}\|_{L_{p',\gamma}(\mathbb{R}^n_{k,+})}\int_0^t \|T^z f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,s)}\,\frac{ds}{s}.$$

Therefore we get that

$$\|T^z M_\gamma f_2\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,t))} \le C\int_0^t \|T^z f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,s))}\,\frac{ds}{s} \le$$

$$\le C\int_0^t \|T^z f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,s))}\frac{ds}{s}. \tag{5}$$

Since

$$\|T^z M_\gamma f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,t))} \le \|T^z M_\gamma f_1\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,t))} + \|T^z M_\gamma f_2\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,t))},$$

from (3) and (5) we arrive at (1) with $\|T^z M_\gamma f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,t))}$ on the left-hand side and then (1) obviously holds also for $\|T^z M_\gamma f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,t))}$. ◄

Let's prove the theorems below.

Theorem 3. *Let* $1 < p < \infty$ *and the functions* ω_1 *and* ω_2 *satisfy condition*

$$\int_0^t s^{-\frac{Q}{p'}} \omega_1(s)\omega_1(s)\frac{ds}{s} \leq Ct^{-\frac{Q}{p'}}\omega_2(t),$$

then the operator M_γ *is bounded from* ${}^{\complement}\mathcal{M}_0^{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ *into* ${}^{\complement}\mathcal{M}_0^{p,\omega_2,\gamma}(\mathbb{R}^n_{k,+})$.

Proof. Let $1 < p < \infty$, $f \in {}^{\complement}\mathcal{M}_0^{p,\theta_1,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$. Hence, by the Theorem 2 we obtain

$$\|M_\gamma f\|_{{}^{\complement}\mathcal{M}_0^{p,\omega_2,\gamma}(\mathbb{R}^n_{k,+})} = \sup_{t>0} \frac{t^{\frac{Q}{p'}}}{\omega_2(t)} \|\chi_{\mathbb{R}^n_{k,+}\setminus E(0,t)} T^{\cdot} M_\gamma f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})} \leq$$

$$\leq C \sup_{t>0} \frac{t^{\frac{Q}{p'}}}{\omega_2(t)} \int_0^t \|T^{\cdot} f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}\setminus E(0,s))} \frac{ds}{s} \leq$$

$$\leq C\|f\|_{\mathcal{M}_0^{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})} \sup_{t>0} \frac{t^{\frac{Q}{p'}}}{\omega_2(t)} \int_0^t s^{-\frac{Q}{p'}} \omega_1(s) \frac{ds}{s} = C\|f\|_{\mathcal{M}_0^{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})}.$$

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