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ON THE ORDER OF APPROXIMATION OF TWO VARIABLE FUNCTIONS BY MELLIN'S DOUBLE SINGULAR INTEGRALS

A.M. MUSAYEV

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In memory of M. G. Gasymov on his 85th birthday

Abstract. *The paper considers the approximation of repeatedly differentiable two variable functions by Mellin's double singular integrals. The order of approximation of repeatedly differentiable functions by means of Mellin's double singular integrals is obtained. The proven theorems apply to specific double singular integrals.*

Keywords: order of approximation, differentiable functions, Mellin's double singular integrals

Mathematics Subject Classification (2020): 26B99, 41A35

1. Introduction

Many questions about the order of approximation and their order of convergence of various classes of functions by linear operators, in particular singular integrals, were studied in the works [1]-[6], [11], [12] and others.

Approximations of functions by singular integrals have numerous applications in various fields of mathematics. Approximations of functions by singular integrals are studied intensively along with other issues in theory of functions. In their papers P.L. Butzer and R.G. Mamedov study convergence order of singular integrals in generating functions at separate characteristic points and metric in the space L^p on bounded and unbounded domains. Important theorems on asymptotic value of approximation of functions by singular integrals are obtained in these papers.

This question for the one-dimensional Mellin singular integral was studied in the work [6]-[10].

A general method for obtaining the asymptotic value of approximating functions by linear positive operators is contained in the monograph [6].

Ali M. Musayev
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
E-mail: emus1957@mail.ru

2. Main Results

Let $f\left(r_1, r_2\right)$ be a real function defined and measurable on $R_2^+ = \begin{pmatrix} 0, & +\infty \\ 0, & +\infty \end{pmatrix}$.

Let us denote $L_p(R_2^+)$ Biggl the space of functions $f\left(r_1, r_2\right)$ for which the norm

$$\|f\|_{L_p\left(R_2^+\right)} = \begin{cases} \iint |f(r_1, r_2)| \; \frac{dr_1}{r_1}\frac{dr_2}{r_2} \quad for \quad p=1, \\ \\ \sup\limits_{(r_1,r_2)\in R_2^+} vrai \quad |f(r_1, \quad r_2)| \quad for \quad p=\infty \end{cases}$$

finite.

Let's denote

$$\Delta_m^k f\left(r_1, \; r_2\right) = \begin{cases} f(r_1, \quad r_2) \;, \quad k=0, \\ \\ \sup\limits_{(r_1,r_2)\in R_2^+} vrai \quad |f(r_1, r_2)| \;, \; k \geq 1. \end{cases}$$

and

$$E^{k_1+k_2}_{r_1^{k_1}, \quad r_2^{k_2}} \quad f\left(r_1, r_2\right) = \begin{cases} f(r, r_2), \quad k_1+k_2=0, \\ E^{(k_2)}_{r_2^{k_2}} \; f(r_1, r_2), \quad k_1=0, \\ E^{(k_1)}_{r_1^{k_1}} \; f(r_1, r_2), \quad k_2=0, \\ E^{(1)}_{r_2} E^{(k_1+k_2-1)}_{r_1^{k_1}, r_2^{k_2-1}} f(r_1, r_2), \quad k_1+k_2 \geq 1. \end{cases}$$

Let us consider Mellin's double singular integral. Let $K_{\lambda_1,\lambda_2}(\rho_1,\rho_2)$ be a function defined on R^+ depending on the real parameters λ_1, λ_2 and biggl and satisfying condition

$$\iint_{R^+} K_{\lambda_1,\lambda_2}(\rho_1,\rho_2)\frac{d\rho_1}{\rho_1}\frac{d\rho_2}{\rho_2} = 1.$$

Expression

$$A_{\lambda_1,\lambda_2}(f;\rho_1,\rho_2) = \iint_{R^+} f\left(\frac{r_1}{\rho_1}, \frac{r_2}{\rho_2}\right) K_{\lambda,\lambda_2}(\rho_1,\rho_2)\frac{d\rho_1}{\rho_1}\frac{d\rho_2}{\rho_2} \tag{1}$$

we will call it Mellin's double singular integral.

Theorem. *Let the function $f\left(r_1, r_2\right) \in L(R^+)$ in a neighborhood of the point r_1, r_2 have continuous derivatives of the n-th order (n-is an even number) and a non-negative function $K_{\lambda_1,\lambda_2}(\rho_1,\rho_2)$ satisfies the conditions:*

a) $K_{\lambda_1, \; \lambda_2}\left(\rho_1,\rho_2\right) = \; K_{\lambda_1, \; \lambda_2}\left(\rho_1^{-1},\rho_2\right) = \; K_{\lambda_1, \; \lambda_2}\left(\rho_1,\rho_2^{-1}\right) = K_{\lambda_1, \; \lambda_2}\left(\rho_1^{-1},\rho_2^{-1}\right)$

b) $\nu^{[2(k-s),2s]}_{\lambda_1\lambda_2} = \iint\limits_{R_2^+} \ln^{2(k-s)}\rho_1 \; . \; ln\rho_2 \; K_{\lambda_1, \; \lambda_2}\left(\rho_1 \; ,\rho_2\right)\frac{d\rho_1}{\rho_1} \quad \frac{d\rho_2}{\rho_2} < +\infty$

at $s \leq k, \; k=1,2,\ldots, \; n/2,$ *where* $R_2^+ = \begin{pmatrix} 1, + \infty \\ 1, \; +\infty \end{pmatrix}$.

If the conditions are met

$$\lim_{\lambda_i\to\lambda_i^0}\frac{\nu_{\lambda_1,\lambda_2}^{(n+j)}}{\nu_{\lambda_1,\lambda_2}^{(n)}}=0 \tag{2}$$

at least for one value $j(j=1,2,\ldots)$, *then the following asymptotic equality holds*

$$A_{\lambda_1,\ \lambda_2}(f;r_1,r_2)-\ f(r_1,r_2)=$$

$$=4\sum_{k=1}^{\left[\frac{n}{2}\right]}\frac{1}{(2k)!}\sum_{s=0}^{k}C_{2K}^{2S}\nu_{\lambda_1,\lambda_2}^{[2(k-s),2s]}.\ E_{r_1^{2(k-s)}r_2^{2s}}^{(2k)}\ f(r_1,r_2)+0\left[\nu_{\lambda_1,\lambda_2}^{(n)}\right] \tag{3}$$

at $\lambda_i\to\lambda_i^0\ (i=1,2)$, *where*

$$\nu_{\lambda_1\lambda_2}^{(n\ )}=\iint_{R_2^+}K_{\lambda_1,\ \lambda_2}(\rho_1,\rho_2)\,\frac{d\rho_1}{\rho_1}\ \ \frac{d\rho_2}{\rho_2}.$$

Proof. Because

$$A_{\lambda_1,\lambda_2,}\ (f;\rho_1,\rho_2)-f(r_1,r_2)=\iint_{R_2^+}\left[f\left(\frac{r_1}{\rho_1},\frac{r_2}{\rho_2}\right)+f\left(r_1\rho_1,\frac{r_2}{\rho_2}\right)+\right.$$

$$\left.+f\left(\frac{r_1}{\rho_1},\ r_2\rho_2\right)+f(r_1\rho_1,\ r_2\rho_2)--4\ f(r_1,r_2)\right]K_{\lambda_1,\ \lambda_2}(\rho_1,\rho_2)\,\frac{d\rho_1}{\rho_1}\ \ \frac{d\rho_2}{\rho_2},$$

then for any $\delta>0$ we have

$$A_{\lambda_1,\ \lambda_2}(f;r_1,r_2)-\ f(r_1,r_2)-4\sum_{k=1}^{\left[\frac{n}{2}\right]}\frac{1}{(2k)!}\sum_{s=0}^{k}C_{2K}^{2S}\nu_{\lambda_1,\lambda_2}^{[2(k-s),2s]}.\ E_{r_1^{2(k-s)},r_2^{2s}}^{(2k)}\ f(r_1,r_2)=$$

$$=\left\{\int_1^{1+}\int_1^{1+}+\int_1^{1+}\int_{1+}^{\infty}+\int_{1+}^{+\infty}\int_1^{1+}+\int_{1+}^{+\infty}\int_{1+}^{+\infty}\right\}\times$$

$$\times\left\{f\left(\frac{r_1}{\rho_1},\frac{r_2}{\rho_2}\right)+f\left(r_1\rho_1,\frac{r_2}{\rho_2}\right)+f\left(\frac{r_1}{\rho_1},\ r_2\rho_2\right)+f(r_1\rho_1,\ r_2\rho_2)-\right.$$

$$\left.-4\sum_{k=1}^{\left[\frac{n}{2}\right]}\frac{1}{(2k)!}\sum_{s=0}^{k}C_{2K}^{2S}\ln^{2(k-s)}\rho_1,\ln^{2s}\rho_2.\ E_{r_1^{2(k-s)}r_2^{2s}}^{(2k)}\ f(r_1,r_2)\right\}$$

$$K_{\lambda_1,\lambda_2}(\rho_1,\rho_2)\,\frac{d\rho_1}{\rho_1}\ \ \frac{d\rho_2}{\rho_2}=\ A_1+A_2+A_3+A_4 \tag{4}$$

By virtue of the conditions of the theorem, it has the equality

$$\varphi_{\rho_1,\ \rho_2}(f;r_1,r_2)=\ 4\sum_{k=1}^{\left[\frac{n}{2}\right]}\frac{1}{(2k)!}\sum_{s=0}^{k}C_{2K}^{2S}\ln^{2(k-s)}\rho_1,\ln^{2s}\rho_2.\ E_{r_1^{2(k-s)},r_2^{2s}}^{(2k)}\ f(r_1,r_2)+$$

$$+0\left[\left(\sqrt{\ln^2\rho_1+\ln^2\rho_2}\right)^n\right]$$

at $\rho_i\to 1+,\quad (i=1,2)$ where

$$\varphi_{\rho_1,\ \rho_2}f(r_1,r_2)=f\left(\frac{r_1}{\rho_1},\frac{r_2}{\rho_2}\right)+f\left(r_1\rho_1,\frac{r_2}{\rho_2}\right)+f\left(\frac{r_1}{\rho_1},\ r_2\rho_2\right)+f(r_1\rho_1,\ r_2\rho_2)$$

Therefore, for $\forall\varepsilon>0$, $\exists\rho>0$ such that for $0<\rho_i-1<(i=1,2)$ bigll $(i=1,2)$ inequality is fair

$$|A_1|<\varepsilon\int_1^{1+}\int_1^{1+}\left(\sqrt{\ln^2\rho_1+\ln^2\rho_2}\right)^nK_{\lambda_1,\ \lambda_2}(\rho_1,\rho_2)\frac{d\rho_1}{\rho_1}\ \ \frac{d\rho_2}{\rho_2}<\varepsilon\nu^n_{\lambda_1,\lambda_2} \tag{5}$$

$A_1=0(\nu^n_{\lambda_1,\lambda_2})$ for $\lambda_i\to\lambda_i^0(i=1,2)$

Let's consider A_2 :

$$|A_2|\le\int_1^{1+}\int_{1+}^{+\infty}\left|\nu_{\lambda_1,\ \lambda_2}(f;r_1r_2)\right|K_{\lambda_1,\ \lambda_2}(\rho_1\rho_2)\frac{d\rho_1}{\rho_1}\ \ \frac{d\rho_2}{\rho_2}+$$

$$+\sum_{k=1}^{\left[\frac{n}{2}\right]}\frac{1}{(2k)!}.\sum_{s=0}^{k}C_{2K}^{2S}\left|E^{(2k)}_{r_1^{2(k-s)}r_2^{2s}}\ f(r_1,r_2)\ \right|.$$

$$\int_1^{1+S}\int_{1+s}^{+\infty}\ln^{2(k-s)}\rho_1,\ln^{2s}\rho_2K_{\lambda_1,\ \lambda_2}(\rho_1\rho_2)\frac{d\rho_1}{\rho_1}\ \ \frac{d\rho_2}{\rho_2}=A_2^{(1)}+A_2^{(2)}.$$

Because

$$A_2^{(1)}\ C_1\int_1^{1+}\int_{1+}^{+\infty}\left|f\left(r_1\rho_1,\frac{r_2}{\rho_2}\right)+f\left(\frac{r_1}{\rho_1},\ r_2\rho_2\right)+\right.$$

$$\left.+f(r_1\rho_1,\ r_2\rho_2)\right|\left(\sqrt{\ln^2\rho_1+\ln^2\rho_2}\right)^{n+j}K_{\lambda_1,\lambda_2}(\rho_1,\rho_2)\frac{d\rho_1}{\rho_1}\ \ \frac{d\rho_2}{\rho_2}\le C_1\|f\|_{L_p}\nu^{n+j}_{\lambda_1\lambda_2}$$

$$A_2^{(2)}\sum_{k=1}^{\left[\frac{n}{2}\right]}\frac{1}{(2k)!}.\sum_{s=0}^{k}C_{2K}^{2S}\left|E^{(2k)}_{r_1^{2(k-s)}r_2^{2s}}\ f(r_1,r_2)\ \right|$$

$$\ln^{(n-2k+j)}(1+\rho)\int_1^{1+S}\int_{1+s}^{+\infty}\left(\sqrt{\ln^2\rho_1+\ln^2\rho_2}\right)^{n+J}K_{\lambda_1,\lambda_2}(\rho_1,\rho_2)\frac{d\rho_1}{\rho_1}\ \ \frac{d\rho_2}{\rho_2}\le C_2\nu^{n+j}_{\lambda_1,\lambda_2}$$

for $j\ge 1$, then by virtue of condition (2) we obtain

$$A_2=0\left(\nu^n_{\lambda_2}\right),\quad \lambda_i\to\lambda_i^0(i=1,2) \tag{6}$$

A_3 and A_4 are evaluated similarly

$$A_3=0\ (\nu^n_{\lambda_2}),\quad A_4=0\left(\nu^n_{\lambda_1,\lambda_2}\right) \tag{7}$$

for $\lambda_i\to\lambda_i^0(i=1,2)$.

From estimates (5), (6), (7), (8) and expression (4) equality (3) follows.

Now we apply the proven theorems to specific Mellin double singular integrals.

Expression

$$\prod_{\lambda_1,\lambda_2}(f;\ r_1,r_2)=\frac{\lambda_1\lambda_2}{4}\iint_{R_2^+} f(\frac{r_1}{\rho_1}\ ,\ \frac{r_2}{\rho_2})e^{-\lambda|\ln\rho_1|-\ \lambda_2|\ln\rho_2|}\frac{d\rho_1}{\rho_1}\frac{d\rho_2}{\rho_2}$$

let's call it the double Mellin integral. In this case

$$\nu_{\lambda_1,\lambda_2}^{[2(k-s,\ 2s)]}=[2(k-s)]!(2s)!,\quad \nu_{\lambda_1,\lambda_2}^{[n]}=\frac{\pi}{2}\frac{\Gamma(n+3)}{\delta^{(n+2)}}<+\infty,$$

where $\delta=(\xi)=cos\xi+\sin\xi$, $\xi\in\left[0,\ \frac{\pi}{2}\right]$.

Therefore, based on the theorem it has

$$\prod_{\lambda_1,\lambda_2}(f;\ r_1,r_2)-f\,(r_1,r_2)=\ 4\sum_{k=1}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{k}\frac{E^{(2k)}_{r_1^{2(k-s)},r_2^{2s}}\ f\,(r_1,r_2)}{\lambda_1^{2(x-s)}\cdot\ \lambda_2^{2s}}+0\ \left(\left[\lambda_1^{-2}+\ \lambda_2^{-2}\right]\right)$$

at $\lambda_i\to\infty(i=1,2)$. ◄

Consequence. *If $f(r_1,r_2)\in L({R_2}^+)$ in the neighborhood of points (r_1,r_2) has continuous partial derivatives of the second order, then the following asymptotic equality holds*

$$\prod_{\lambda_1,\lambda_2}(f;\ r_1,r_2)-f\,(r_1,r_2)=4\left[\frac{E^{(2)}_{r_1^2}\ f\,(r_1,r_2)}{\lambda_1^2}+\ \frac{E^{(2)}_{r_2^2}\ f\,(r_1,r_2)}{\lambda_2^2}\right]+0(\lambda_1^{-2}+\lambda_2^{-2})$$

at $\lambda_i\to\infty(i=1,2)$.

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