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ON COMBINING THE STAGES OF PARAMETRIC IDENTIFICATION AND OPTIMIZATION OF DYNAMIC PROCESSES

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In memory of M. G. Gasymov on his 85th birthday

Abstract. *In the paper, we propose a method for jointly solving problems of parametric identification of a mathematical model's parameters and searching for the optimal mode of a dynamic object. As a result of this joint process, we obtain a suboptimal solution to the problem of controlling a dynamic object in the vicinity of the optimal mode. We present computation formulas and algorithms for implementing the proposed approach, as well as the results of computer-based experiments.*

Keywords: dynamic object, parametric identification, optimal control, lumped systems, mathematical modelling

Mathematics Subject Classification (2020): 34A34, 90C30, 90C31, 93-10

1. Introduction

Effective control of technological processes and technical objects depends to a large extent on the quality of mathematical models that describe these processes. To study technological processes and control them, the statistical modelling apparatus is widely used, which has proven itself well provided that the drift of mathematical models is sufficiently small. In the case of complex, essentially nonlinear, dynamic processes, it is necessary to solve the problem of selecting an adequate mathematical model. Since a model that is sufficiently adequate on average for the entire range of permissible process parameters is not

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always sufficiently satisfactory for any selected local area, it is proposed to use parametric identification of the mathematical model of the process with its sequential refinement, which is carried out when optimizing process regimes. Thus, for decision-making systems on controlling dynamic processes, including the stages of parametric identification of a mathematical model and optimization of regimes, we propose an approach in which these two stages are carried out jointly (alternately). The result of this joint process is obtaining a "locally optimal" model in the vicinity of an optimal regime. This paper derives calculation formulas and an algorithm for implementing the proposed approach.

The main advantage of the proposed approach is that the adopted solution is optimal for the mathematical model built taking into account the observations of the information model under consideration, which are closest to the optimal solution, i.e., the mathematical model of the object can be considered locally optimal with respect to the decision made. In contrast to the classical two-stage approach to decision making, the proposed approach requires a larger amount of computations due to the need to carry out parametric identification of the model at each iteration of the optimization process. We describe two variants of the decision-making procedure, which are iterative processes that combine optimization of control parameters with redefinition of the parameters of the mathematical model. The first option is that at each iteration of optimization of control parameters and identification of the model parameters, we completely cut off non-local observable information. The second option involves using the entire set of observable data, but each observation receives a weight depending on the distance of this observation from the computed current optimum point obtained for the model built in the previous step.

The results of numerical experiments confirm the effectiveness of using the approach proposed in the work to solve the problems. It is also necessary to take into account that the optimal controls obtained during the solution may differ significantly from the controls obtained using the classical sequential two-stage approach to parametric identification and optimization of regimes.

2. Problem Statement

Let the current state $x(t) \in \mathbb{R}^n$ of a controllable technological process or a dynamic object be determined by the initial state $x_0 \in \mathbb{R}^n$, the vector of nonadjustable (uncontrollable) parameters $v \in V \subseteq \mathbb{R}^m$ with values set at the beginning of the process, and the vector of adjustable (controllable) parameters $u \in U \subseteq \mathbb{R}^r$ with values assigned over the entire process period $t \in [0, T]$, subject to minimization of a given functional, which determines the criterion for optimal process control. Let us assume that the process is described by an initial-value problem with respect to some autonomous system of nonlinear differential equations of the form

$$\begin{gathered} \dot{\widetilde{x}}(t) = \widetilde{f}\left(\widetilde{x}(t), u, v, p\right), \quad t \in [0, T] \\ \widetilde{x}(0) = \widetilde{x}_0 \in X_0 \end{gathered} \tag{1}$$

where X_0 is the set of possible initial states of the process; $\widetilde{x}(t) = \widetilde{x}\left(t; \widetilde{x}_0, u, v\right)$ the function that defines the process state under given initial conditions $\widetilde{x}_0$, values of the nonadjustable parameters $v \in V$ determined at the beginning of its course, and assigned

(selected) values of adjustable parameters $u \in U$; $p \in P \subseteq \mathbb{R}^l$ the vector of parameters of the mathematical model, the values of which belong to the set of permissible values of the parameters of the mathematical model of the process, P; U the set of permissible values of control parameters; V the set of possible values of nonadjustable parameters.

Let the functional

$$\mathcal{J}\left(u; \widetilde{x}_0, v, p\right) = \int_0^T f^0\left(\widetilde{x}(t), u, v\right) dt + \Phi\left(\widetilde{x}(T), u, v\right) \to \min_{u \in U} \tag{2}$$

determine the quality of the selected values of the vector of control parameters u depending on the specified initial state $\widetilde{x}_0$ and the values of the vector of nonadjustable parameters v. Here $f^0(\cdot)$ and $\Phi(\cdot)$ are given functions continuously differentiable with respect to the their first two arguments.

When designing and developing control systems for real technological processes and technical objects, the vector function $\widetilde{f}$, which determines the dynamics of the process, is usually not specified, or only the class of functions is specified up to parameters that require estimating values. The problem of reconstructing the function $\widetilde{f}(\cdot)$ belongs to the stage of mathematical modelling and for its implementation it is necessary to have the results of observations of the process. This stage consists of two sub-stages. At the first sub-stage (called "structural identification"), one defines a class of functions that depends on the parameters p, the values of which are determined at the second sub-stage (called "parametric identification") [4].

We assume that the first sub-stage of process modelling – the problem of structural identification – has been already resolved, for example, by means of some a priori qualitative information about the nature of the process, i.e., the process, instead of (1), is described by the following system of differential equations:

$$\dot{x}(t) = f\left(x(t), u, v, p\right), \quad t \in [0, T], \tag{3}$$

where $f(\cdot)$ is an n-dimensional vector function specified up to parameters p, continuously differentiable with respect to its arguments, which most often differs from the function $\widetilde{f}(\cdot)$ that actually describes the process; p the vector of parameters of the mathematical model of the process, the values of which need to be determined at the parametric identification sub-stage.

For parametric identification, it is necessary to conduct observations of the state of the technological process, which can be of a different nature [1]. For example, for given values of nonadjustable and controllable parameters $v^i \in V$ and $u^i \in U$, $i = 1, 2, \ldots, N$, respectively, observations of the process state can be conducted at given moments of time $t_{i,j} \in [0, T]$,

$$\widehat{x}^i_j = \widehat{x}^i\left(t_{i,j}; u^i, v^i\right), \quad t_{i,j} \in [0, T], \quad j = 0, 1, \ldots, M_i, \quad i = 1, 2, \ldots, N. \tag{4}$$

or only at the initial and final moments of time, $t_{i,0} = 0$ and $t_{i,M_i} = T$:

$$\widehat{x}^i_0 = \widehat{x}^i\left(0; u^i, v^i\right), \quad \widehat{x}^i_T = \widehat{x}^i\left(T; u^i, v^i\right), \quad i = 1, 2, \ldots, N, \tag{5}$$

where N is the number of observations of separately occurring processes; M_i the number of observations of the state during each process; for each of the observations, a positive weight coefficient $\gamma_i \in [0,1]$, $i = 1,2,\ldots,N$, is specified, the values of which are determined by the degree of reliability and accuracy of the results of observations.

Observations can also be conducted at specified time intervals:

$$\widehat{x}_j^i(t) = \widehat{x}^i\left(t; u^i, v^i\right), \quad t \in [t_{i,j-1}, t_{i,j}], \quad j = 0,1,\ldots,M_i, \quad i = 1,2,\ldots,N, \tag{6}$$

where $[t_{i,j-1}, t_{i,j}] \subset [0,T]$ are the specified intervals; M_i is the number of observable intervals for each observation.

The problem of identifying the model parameters p (parametric identification problem) using, for example, the least squares criterion, depending on the type of observations leads to minimization of the functional [6]

$$\mathcal{S}_1(p) = \sum_{i=1}^{N}\sum_{j=1}^{M_i} \gamma_i \left\| x^i\left(t_{i,j}; u^i, v^i, p\right) - \widehat{x}_j^i \right\|_{\mathbb{R}^n}^2 + \epsilon \|p\|_{\mathbb{R}^l}^2 \tag{7}$$

in the case of observations of type (4); for observations of type (5), instead of functional (7), we consider the following functional:

$$\mathcal{S}_2(p) = \sum_{i=1}^{N} \gamma_i \left\| x^i\left(T; u^i, v^i, p\right) - \widehat{x}_T^i \right\|_{\mathbb{R}^n}^2 + \epsilon \|p\|_{\mathbb{R}^l}^2 . \tag{8}$$

For observations of type (6), the following functional is taken:

$$\mathcal{S}_3(p) = \sum_{i=1}^{N}\sum_{j=1}^{M_i} \gamma_i \int_{t_{i,j-1}}^{t_{i,j}} \left\| x^i\left(t; u^i, v^i, p\right) - \widehat{x}_j^i(t) \right\|_{\mathbb{R}^n}^2 dt + \epsilon \|p\|_{\mathbb{R}^l}^2, \tag{9}$$

where $x^i\left(t; u^i, v^i, p\right)$ is the solution of the system of differential equations (3) under initial conditions

$$x^i\left(0; u^i, v^i, p\right) = \widehat{x}_0^i \tag{10}$$

(i.e., for $t_{i,0} = 0$) and for given values of the parameters u^i, v^i, and p; ϵ the regularization parameter of the minimizable functional [8].

Thus, the problem of creating a mathematical software for controlling the given technological process consists of two stages. At the first stage, we solve the problem of parametric identification of the mathematical model of the process, described by (3) and (10), using the corresponding observations (4), (5), or (6) by minimizing one of the functionals (7), (8), or (9); at the second stage, we solve the problem (2) and (3) on choosing an optimal mode $\overline{u} = \overline{u}\,(t; \overline{x}_0, \overline{v}) = const$ of the process under given values of the initial state $\overline{x}_0$ and the values of the vector of nonadjustable parameters $\overline{v}$.

3. Numerical Solution of Parametric Identification and Control Problems

It is clear that the formulation of problems of parametric identification (3)-(9) and of optimal control (2) and (3) belong to the same class of parametric optimal control problems; in order to solve them, any known approach can be used; in particular, we can apply first order numerical optimization methods. For example, in the case of a simple structure of the admissible regions of parameters, P and U (i.e., multidimensional parallelepiped, sphere, etc.), the gradient projection method can be used [3], [9]. To solve the parametric identification problem (3)-(9), we build an iterative process using the following formulas:

$$p^{k+1} = Proj_P\left(p^k - \alpha_k \times \nabla\mathcal{S}\left(p^k\right)\right), \quad k = 0, 1, 2, \ldots, \tag{11}$$

$$\nabla\mathcal{S}\left(p^k\right) = 2\epsilon\, p^k - \sum_{i=1}^{N} \int_0^T \frac{\partial f\left(x^i(t), u^i, v^i, p^k\right)}{\partial p} \cdot \psi^i(t)dt, \tag{12}$$

where $Proj_P(\cdot)$ is the projection operator onto the admissible domain P, which has a simple form for a domain of simple structure; $x^i(t) = x^i\left(t; \widehat{x}_0^i, u^i, v^i\right)$ the solution of the initial-value problem (3) and (5); α_k the one-dimensional minimization step; $\psi^i(t)$ the solution of an adjoint problem, the form of which depends on the observations conducted and the selected objective functional. For example, for observations (4) and (5) and functional (7), the adjoint system is as follows:

$$\begin{gathered}\dot{\psi}^i(t) = -\left[\frac{\partial f\left(x^i(t), u^i, v^i, p\right)}{\partial x}\right]^* \psi^i(t) + \\ +2\gamma_i \sum_{j=1}^{M_i} \left[x^i\left(t; u^i, v^i, p\right) - \widehat{x}_j^i\right] \cdot \delta\left(t - t_{i,j}\right), \quad t \in (0, T],\end{gathered} \tag{13}$$

$$\psi^i(T) = 0, \quad i = 1, 2, \ldots, N, \tag{14}$$

where $\delta\left(t - t_{i,j}\right)$ is the generalized Dirac function; $*$ the matrix transpose sign. For observations of type (6) and functional (9), the adjoint system takes the form

$$\begin{gathered}\dot{\psi}^i(t) = -\left[\frac{\partial f\left(x^i(t), u^i, v^i, p\right)}{\partial x}\right]^* \psi^i(t) + \\ +2 \sum_{j=1}^{N} \chi_j^i(t) \ \left[x^i\left(t; u^i, v^i, p\right) - \widehat{x}^i(t)\right], \quad t \in (0, T],\end{gathered} \tag{15}$$

with initial condition (14), where $\chi_j^i(t)$ is the Heaviside type function defined in our case as follows:

$$\chi_j^i(t) = \begin{cases} 0,\ t \notin \bigcup\limits_{j=1}^{N} [t_{i,j-1}, t_{i,j}] \bigcap [t, T], \\ 1,\ t \in \bigcup\limits_{j=1}^{N} [t_{i,j-1}, t_{i,j}] \bigcap [t, T]. \end{cases} \tag{16}$$

The above formulas can be easily derived using the method of variations of optimizable parameters or the corresponding well-known formulas [3], [5], [9], taking into account that problems (3)-(9) are a special case of the general formulation of the optimal control problem.

After solving the parametric identification problem and determining the parameters p with a given accuracy, and consequently, constructing a mathematical model of the process, to solve the optimal control problem for a given initial condition $\overline{x}^0$ and the value of the nonadjustable parameter $\overline{v}$, first-order numerical optimization methods can be used [3], [9]:

$$u^{k+1} = Proj_U\left(u^k - \alpha_k \times \nabla\mathcal{J}\left(u^k\right)\right), \quad k = 0, 1, 2, \ldots, \tag{17}$$

$$\nabla\mathcal{J}\left(u^k\right) = \int_0^T \left\{ -\frac{\partial f^0\left(x^k(t), u^k, \overline{v}, p\right)}{\partial u} + \left[\frac{\partial f\left(x^k(t), u^k, \overline{v}, p\right)}{\partial u}\right]^* \xi^k(t) \right\} dt + \frac{\partial \Phi\left(x^k(T), u^k, \overline{v}\right)}{\partial u}. \tag{18}$$

Here $Proj_U(\ )$ is the projection operator onto the admissible domain U; $x^k(t) = x\left(t; \overline{x}^0, u^k, \overline{v}, p\right)$ the solution of the initial-value problem with respect to (3); α_k the step in the direction of the anti-gradient of the functional in the space of control parameters; $\xi^k(t) = \xi\left(t; \overline{x}^0, u^k, \overline{v}\right)$ the solution of the following adjoint problem:

$$\dot{\xi}^k(t) = \frac{\partial f^0\left(x^k(t), u^k, \overline{v}, p\right)}{\partial x} - \left[\frac{\partial f\left(x^k(t), u^k, \overline{v}, p\right)}{\partial x}\right]^* \xi^k(t), \quad t \in (0, T], \tag{19}$$

$$\xi^k(T) = -\frac{\partial \Phi\left(x^k(T), u^k, \overline{v}\right)}{\partial x}. \tag{20}$$

The disadvantage of the described two-stage approach to decision-making on process control, in which at the first stage a mathematical model is built using all the available observations, is the following. The right-hand sides of differential equations (1), describing the process under consideration, as a rule, are not known exactly, and when mathematically modelling the process, i.e., when choosing the function $f(\cdot)$ in (3), simplified (linear, quadratic) functions are used first of all. Therefore, after parametric identification of the mathematical model, the calculated trajectory values will not coincide with the real trajectory. The greater the spread of observable admissible values of the initial conditions $\widehat{x}_0^i$, control parameters u^i, and nonadjustable parameters v^i, the more the calculated values of the mathematical model of the process will differ from the real trajectory.

To develop the results obtained in [2], [7] regarding static objects, it is proposed to combine the stages of mathematical modelling, more precisely, the stages of parametric identification of parameters p and of optimization of control parameters u, but only after the values of the initial conditions $\overline{x}^0$ and nonadjustable parameters $\overline{v}$ are specified. Parametric identification of the model should be carried out after each iteration (17) on optimization of the control parameters u. It is very important that all available observations used for the parametric identification problem be assigned some weights $\rho^i\left(\widehat{x}_0^i, v^i, u^i; \overline{x}^0, \overline{v}, u^k\right)$, $i = 1, 2, \ldots, N$, which are inversely proportional to the distance between the observable parameters $\widehat{x}_0^i, v^i, u^i$ of the process, and the given values of the initial condition $\overline{x}^0$, the nonadjustable parameter $\overline{v}$, and the current value of the iterative process (17) on optimization of the control parameter u^k. In this case, for example,

functional (7) used in the parametric identification problem, at the k^{th} iteration of the process of optimizing the control parameter u^k will have the form

$$\mathcal{S}_1^k\left(p;\overline{x}^0,\overline{v},u^k\right)=\sum_{i=1}^{N}\rho^i\left(\overline{x}^0,\overline{v},u^k\right)\sum_{j=1}^{M_i}\gamma_i\left\|x^i\left(t_{i,j};u^i,v^i,p\right)-\widehat{x}_j^i\right\|_{\mathbb{R}^n}^2+\epsilon\left\|p\right\|_{\mathbb{R}^l}^2, \tag{21}$$

and, when $M_i=1$ and $t_{i,M_i}=T$, the functional (8) takes the form

$$S_2^k\left(p;\overline{x}^0,\overline{v},u^k\right)=\sum_{i=1}^{N}\rho^i\left(\overline{x}^0,\overline{v},u^k\right)\ \gamma_i\left\|x^i\left(T;u^i,v^i,p\right)-\widehat{x}_T^i\right\|_{\mathbb{R}^n}^2+\epsilon\left\|p\right\|_{\mathbb{R}^l}^2. \tag{22}$$

Functional (9) will change similarly. As weight functions, one can use, for example, a function of the form

$$\rho^i\left(\overline{x}^0,\overline{v},u^k\right)=\rho_x^i\left(\overline{x}^0\right)+\rho_v^i\left(\overline{v}\right)+\rho_u^i\left(u^k\right),$$

$$\rho_x^i\left(\overline{x}^0\right)=e^{-\theta_1\left\|\widehat{x}_0^i-\overline{x}^0\right\|};\qquad \rho_v^i\left(\overline{v}\right)=e^{-\theta_2\left\|v^i-\overline{v}\right\|};\qquad \rho_u^i\left(u^k\right)=e^{-\theta_3\left\|u^i-u^k\right\|}, \tag{23}$$

where θ_1, θ_2, and θ_3 are some predefined constant coefficients. It is evident that the values $\rho_x^i\left(\overline{x}^0\right)$, $\rho_v^i\left(\overline{v}\right)$, and $\rho_u^i\left(u^k\right)$ will not change during the decision-making process (both for the parametric identification and the optimal control problems). They are defined as soon as the values of the initial condition $\overline{x}^0$ and the nonadjustable parameters $\overline{v}$ become known. In the iterative optimization process (17), the parameters of the mathematical model p will change due to changes in $\rho_u^i\left(u^k\right)$, $i=1,2,\ldots,N$, $k=1,2,3,\ldots$.

Remark 1. Instead of (23), one can also use the product of the weight functions:

$$\rho^i\left(\widehat{x}_0^i,v^i,u^i;\overline{x}^0,\overline{v},u^k\right)=\rho_x^i\left(\overline{x}^0\right)\times\rho_v^i\left(\overline{v}\right)\times\rho_u^i\left(u^k\right)$$

or any other function with the property of monotonic decrease depending on the value of the modules of the difference of the arguments $\left\|\widehat{x}_0^i-\overline{x}^0\right\|$, $\left\|v^i-\overline{v}\right\|$, and $\left\|u^i-u^k\right\|$.

Remark 2. Another implementation of the proposed approach to combining the solution of the problems of parametric identification and optimal control can be cutting off from the set of observable values of the process parameters those observations whose values are away from the current vector $(\overline{x}^0,\overline{v},u^k)$ at a distance greater than some predefined quantity. In this case, the weight functions $\rho^i(\cdot)$ are not used; at each optimization iteration (17), parametric identification is carried out using only a truncated set of observations [2].

In the proposed approach to mathematical modelling and decision-making on process control, the problems of parametric identification and of optimal control are replaced by a single task, which can be written in the form

$$\mathcal{J}\left(u;\overline{x}^0,\overline{v},\arg\min_{p\in P}\mathcal{S}\left(p;\overline{x}^0,\overline{v},u\right)\right)\to\min_{u\in U} \tag{24}$$

or

$$\mathcal{J}\left(u;\overline{x}^0,\overline{v},p^*(u)\right)\to\min_{u\in U},\quad p^*(u)=\arg\min_{p\in P}\mathcal{S}\left(p;\overline{x}^0,\overline{v},u\right) \tag{25}$$

taking into account differential equations (3). The values $(\overline{x}^0, \overline{v})$ in (24) and (25) are given, the functionals $\mathcal{J}(\cdot)$ and $\mathcal{S}(\cdot)$ are defined by formulas of the kind (2) and (21) or (22). An optimal pair $(u^*, p^*(u))$, which is a solution to the problem (24) or (25), is characterized by the fact that during the parametric identification of the model (3), observations (4)-(6) conducted in closer neighbourhoods of the optimal solution u^* have greater weight, i.e., the mathematical model is "locally optimal" in the neighbourhood of the parameter values $(u^*, p^*(u))$.

Thus, to solve the stated problem, one can use the following algorithm.

Step 1. For given values of the initial state $\overline{x}^0$ and nonadjustable parameter $\overline{v}$, for all available observations (4), (5) or (6), we calculate the values of the weight coefficients $\rho_x^i\left(\overline{x}^0\right)$ and $\rho_v^i\left(\overline{v}\right)$, whereas the values of the weights $\rho_u^i\left(u^k\right)$, $i = 1, 2, \ldots, N$, we take equal to one.

Step 2. Solving the problem (11)-(16), we carry out the parametric identification and determine the vector of model parameters p^{*0}.

Step 3. Using the iterative method (17)-(20), we solve the optimal control problem (2) and (3) on determining the vector u^{*0}.

Step 4. We perform one step of the iterative method (17)-(20), determine a new vector of parameters u^{*1}, and solve the parametric identification problem with recalculated values of the weight coefficients $\rho_u^i\left(u^{*0}\right)$, $i = 1, 2, \ldots, N$.

Step 5. If $\left\|u^{*1} - u^{*0}\right\| > \varepsilon$, then, putting $u^{*0} = u^{*1}$, we go back to step 4; otherwise, the problems of parametric identification and optimal control has been solved with the given accuracy $\varepsilon > 0$.

When using the proposed algorithm in automated process control systems, the need to memorize the mathematical model is replaced by the need to save the so-called "information model" of the process, consisting of the differential equations (3), optimizable functional (2), and observation values for the process state parameters (4), (5), or (6).

It is clear that the amount of calculations in the proposed approach exceeds the amount of calculations when using the classical separate two-stage approach to mathematical modelling and optimal control. But in the proposed mathematical model, in order to make an optimal decision, specifically given values of the initial state and nonadjustable process parameters are taken into account, and in addition, it is locally optimal in the vicinity of $u^* \in U$. However, it should be taken into account that the power of modern computing systems makes it possible to solve decision-making problems on the control of many real technological processes and technical objects using the proposed approach in real time and obtain more accurate optimal solutions.

Remark 3. The proposed approach can be easily extended to the case of discrete dynamic objects described by systems of the form $x_{k+1} = f\left(x_k, u, v, p\right)$, $k = 0, 1, 2, \ldots, M$, for which one poses the problem of parametric identification of parameters $p \in \mathbb{R}^l$ under available observations of the process state at any discrete moments of time and optimization of the control vector of parameters $u \in \mathbb{R}^r$ with respect to any given objective functional to assess the quality of control.

4. Results of Numerical Experiments

Let us consider the results of solving the test problem using the proposed approach. Let us assume that the process under study is described by the following initial-value problem with respect to an autonomous system of nonlinear differential equations of the form

$$\dot{\widetilde{x}}_1(t) = \frac{1}{2}\widetilde{x}_2^2(t) - \widetilde{x}_2(t) - v\ u_1 + 1; \qquad \dot{\widetilde{x}}_2(t) = \widetilde{x}_1(t) + u_1 - 1;$$

$$\dot{\widetilde{x}}_3(t) = -\widetilde{x}_4^2(t) + 2\widetilde{x}_4(t) - 2\widetilde{x}_3(t) + v\ u_2; \qquad \dot{\widetilde{x}}_4(t) = -\widetilde{x}_3(t) - u_2 + 1,$$
$$t \in (0, T], \quad \widetilde{x}_0 \in X_0 = \{\widetilde{x}(0) : -3 \le \widetilde{x}_i(t) \le 3,\ i = 1, 2, 3, 4\} \tag{26}$$

Here $v \in V = [-3, 3] \subset \mathbb{R}$ is the value of the nonadjustable parameter; $u = (u_1, u_2) \in U \equiv \mathbb{R}^2$ the assignable values of the adjustable parameters; $T = 2$ the process operating time. The functional

$$\mathcal{J}(u; \widetilde{x}_0, v, p) = \sum_{i=1}^{4} \left[\widetilde{x}_i(T; \widetilde{x}_0, u, v) - \widetilde{x}_i^T\right]^2 \to \min_{u \in U}, \tag{27}$$

where $\widetilde{x}_T = (2.0589;\ -4.3317;\ -1.9177;\ 1.2578)$, determines the quality of the selected values of the vector of control parameters u under the already definitely specified initial state $\widetilde{x}_0$ and the value of the nonadjustable parameter v. It can be verified numerically that the result of solving the process optimization problem (26) for $x(0) = (2.35;\ 2.35;\ -2.15;\ -2.15)$ and $v = -1.95$ is the vector $u^* = (-2.25; 2.25)$, for which the functional (27) takes on the value $\mathcal{J}^* = 0$.

Let us assume that instead of the process (26), the exact form of which we will consider unspecified, the mathematical model of the process be defined by the following system of differential equations:

$$\dot{x}_1(t) = p_1 x_2^2(t) - v\ u_1; \quad \dot{x}_2(t) = p_2 x_1(t) - p_2 + u_1;$$

$$\dot{x}_3(t) = p_3 x_4^2(t) + p_4 x_3(t) - p_4 + v\ u_2; \quad \dot{x}_4(t) = -x_3(t) - u_2 + 1, \tag{28}$$

where $p = (p_1\ , p_2, p_3, p_4) \in P \equiv \mathbb{R}^4$ is the vector of parameters of the mathematical model. Let, for different given values of the nonadjustable and controllable parameters $v^i \in V$ and $u^i \in U$, $i = 1, 2, \ldots,\ N$, there are results of observations of the state of the technological process (26) at the initial and final moments of time $t_{i,0} = 0$, $t_{i,M} = 2$:

$$\widehat{x}_0^i = \widehat{x}^i\left(0; u^i, v^i\right), \quad \widehat{x}_T^i = \widehat{x}^i\left(T; u^i, v^i\right), \quad i = 1, 2, \ldots, N,$$

where $N = 60$ is the number of observations made of separately occurring processes. Let us write out the observable initial states of the object explicitly as follows

$$\widehat{x}_j^i(0) = -3 + (i-1)\frac{6}{N}, \quad j = 1, 2, 3, 4, \quad i = 1, 2, \ldots, N, \tag{29}$$

under the following values of the controllable and nonadjustable parameters:

$$u_j^i = -3 + (i-1)\frac{6}{N},\ j = 1, 2; \qquad v^i = -3 + (i-1)\frac{6}{N},\ i = 1, 2, \ldots, N.$$

The determination of the parameters p of the model (the parametric identification problem) reduces to the minimization of the following functional:

$$\mathcal{S}(p) = \sum_{i=1}^{N} \gamma_i \left\| x^i\left(T; u^i, v^i, p\right) - \widehat{x}_T^i \right\|_{\mathbb{R}^4}^2 + \epsilon \left\| p \right\|_{\mathbb{R}^4}^2 .$$

Assuming that the accuracy of all the conducted observations is the same, we let $\gamma_i = 1$, $i = 1, 2, \ldots, N$. Under the condition $P \equiv \mathbb{R}^4$, i.e., there are no constraints on the values of the parameters, the iterative process (11) and (12) is determined as follows:

$$p^{k+1} = p^k - \alpha_k \times \nabla \mathcal{S}\left(p^k\right), \quad k = 0, 1, 2, \ldots,$$

$$\nabla \mathcal{S}\left(p^k\right) = \left(2\epsilon\ p_1^k + \sum_{i=1}^{N} \int_0^T -\psi_1^i(t) \left[x_2^i(t)\right]^2 dt;\ \ 2\epsilon\ p_2^k + \sum_{i=1}^{N} \int_0^T -\psi_2^i(t) x_1^i(t) dt; \right.$$

$$\left. 2\epsilon\ p_3^k + \sum_{i=1}^{N} \int_0^T -\psi_3^i(t) \left[x_4^i(t)\right]^2 dt; 2\epsilon\ p_4^k + \sum_{i=1}^{N} \int_0^T \psi_4^i(t) x_3^i(t) dt \right),$$

where $x^i(t) = x^i\left(t; \widehat{x}_0^i, u^i, v^i\right)$ is the solution to the initial-value problem (28) and (29); $\psi^i(t)$ the solution to the following adjoint problem, obtained from (13) and (14):

$$\dot{\psi}_1^i(t) = \psi_2^i(t)\ p_2; \quad \dot{\psi}_2^i(t) = -2\psi_1^i(t)\ p_1\ x_2^i(t);$$

$$\dot{\psi}_3^i(t) = \psi_3^i(t)\ p_4 + \psi_4^i(t); \quad \dot{\psi}_4^i(t) = -2\psi_3^i(t)\ p_3 x_4^i(t), \quad t \in [0, T),$$

$$\psi^i(T) = \left(-2\gamma_i \left[x_1^i\left(T; u^i, v^i, p^k\right) - \widehat{x}_{1,T}^i\right];\ -2\gamma_i \left[x_2^i\left(T; u^i, v^i, p^k\right) - \widehat{x}_{2,T}^i\right];\right.$$

$$\left. -2\gamma_i \left[x_3^i\left(T; u^i, v^i, p^k\right) - \widehat{x}_{3,T}^i\right]; -2\gamma_i \left[x_4^i\left(T; u^i, v^i, p^k\right) - \widehat{x}_{4,T}^i\right]\right), \quad i = 1, 2, \ldots, N.$$

To solve the optimal control problem under the given initial condition $\overline{x}_0$ and the value of the nonadjustable parameter $\overline{v}$, we use the procedure (17) and (18) taking into account that $U \equiv \mathbb{R}^4$:

$$u^{k+1} = u^k - \alpha_k \times \nabla \mathcal{J}\left(u^k\right), \quad k = 0, 1, 2, \ldots,$$

$$\nabla \mathcal{J}\left(u^k\right) = \left(\int_0^T \left[\xi_1(t) v - \xi_2(t)\right] dt; \ \ \int_0^T \left[-\xi_3(t) v + \xi_4(t)\right] dt \right),$$

where $x^k(t) = x^k\left(t; \overline{x}_0, u^k, \overline{v}\right)$ is the solution of the initial-value problem with respect to (29); $\xi^k(t) = \xi^k\left(t; \overline{x}_0, u^k, \overline{v}\right)$ the solution to the adjoint problem:

$$\dot{\xi}_1(t) = \xi_2(t)\ p_2; \quad \dot{\xi}_2(t) = -2\xi_1(t)\ p_1\ x_2(t);$$

$$\dot{\xi}_3(t) = \xi_3(t)\ p_4 + \xi_4(t); \quad \dot{\xi}_4(t) = -2\xi_3(t)\ p_3\ x_4(t), \quad t \in [0, T),$$

$$\xi(T) = \left(-2\left[x_1\left(T; \overline{x}_0, u^k, \overline{v}\right) - \overline{x}_{1,T}\right]; \quad -2\left[x_2\left(T; \overline{x}_0, u^k, \overline{v}\right) - \overline{x}_{2,T}\right]\ ;\right.$$

$$\left. -2\left[x_3\left(T; \overline{x}_0, u^k, \overline{v}\right) - \overline{x}_{3,T}\right]; \quad -2\left[x_4\left(T; \overline{x}_0, u^k, \overline{v}\right) - \overline{x}_{4,T}\right]\right).$$

As a result of numerical solution of the problem using the classical separate two-stage approach to mathematical modeling and optimal control of the process under consideration, the values of the identifiable parameters of the mathematical model were obtained as $\overline{p}^* = (0.6104;\ 1.4495;\ -2.0522;\ -2.0037)$ and optimal values of the corresponding control parameters $\overline{u}^* = (-2.3918;\ 2.4929)$ under optimal values of the objective functionals $\mathcal{S}(\overline{p}^*) \approx 1195.47$ and $\mathcal{J}(\overline{u}^*) \approx 14.53$. As a result of numerical solution of the problem using the proposed approach, the values of the identifiable parameters of the mathematical model were computed as $\overline{p}^* = (0.5148;\ 2.2775;\ -2.2108;\ -1.6383)$ and optimal values of the corresponding control parameters $\overline{u}^* = (-2.1678;\ 2.2651)$ under optimal values of the objective functionals $\mathcal{S}(\overline{p}^*) \approx 7.0 \times 10^{-4}$ and $\mathcal{J}(\overline{u}^*) \approx 7.86$. The accuracy of the solution of the main optimization problem with respect to the optimization of the controllable parameters is $\varepsilon_1^u = 10^{-3}$, and the accuracy of the solution of the auxiliary one-dimensional optimization problem is $\varepsilon_2^u = 10^{-4}$. The accuracy of the solution to the main optimization problem relative to the optimization of the model parameters is $\varepsilon_1^p = 10^{-4}$, and the accuracy of the solution to the auxiliary one-dimensional optimization problem is $\varepsilon_2^p = 10^{-5}$. To numerically solve the initial-value problem for the direct and adjoint systems, the fourth order Runge–Kutta method was employed with the step $h = 0.02$. The values of the coefficients θ_i, $i = 1, 2, 3$, in the expression of the weight functions $\rho_x^j\left(\overline{x}^0\right) = e^{-\theta_1\left\|\widehat{x}_0^j - \overline{x}^0\right\|}$; $\rho_v^j\left(\overline{v}\right) = e^{-\theta_2\left\|v^j - \overline{v}\right\|}$; $\rho_u^j\left(u^k\right) = e^{-\theta_3\left\|u^j - u^k\right\|}$ were taken equal to $\theta_1 = \theta_2 = 1.00$ and $\theta_3 = 2.50$.

When comparing the described models, a fairly significant difference in the values of their parameters is visible, which, in turn, implies a difference in the obtained optimal values of the control parameters. Moreover, the optimal values of the control parameters obtained by the proposed method are closer to the actual optimal vector $u^* = (-2.25;\ 2.25)$ than those obtained by the two-stage method.

5. Conclusion

The main advantage of the proposed approach is that the adopted solution is optimal for the mathematical model constructed taking into account the observations of the information model under consideration, which are closest to the optimal solution, i.e., the mathematical model of the object can be considered locally optimal with respect to the decision made. Unlike the classical two-stage approach to decision making, the proposed approach requires a larger amount of computer calculations due to the need to carry out parametric identification of the model at each iteration of the optimization method. The results of the numerical experiments confirm the effectiveness of the proposed approach to solving the problems. It is necessary to take into account that the optimal controls obtained during the solution may differ significantly from the controls obtained using the classical sequential two-stage approach to parametric identification and optimization.

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