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ON AN INVERSE SCATTERING PROBLEM FOR A SYSTEM OF DIRAC EQUATIONS WITH NONLINEAR DEPENDENCE ON THE SPECTRAL PARAMETER

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In memory of M. G. Gasymov on his 85th birthday

Abstract. *This paper explores an inverse scattering problem for a system of Dirac equations with boundary condition depending on spectral parameter nonlinearly. We provide the scattering data corresponding to the boundary value problem and investigate some properties. In order to discuss inverse problem, the main equation is derived and its unique solvability is proved. As a consequence, we present the reconstruction of the potential matrix function from scattering data uniquely.*

Keywords: scattering data, Dirac equations system, spectral parameter

Mathematics Subject Classification (2020): 34L25, 34L40, 34B07, 81U40

1. Introduction

In 1929, the Dirac equation was discovered by P. Dirac, which describes the relativistic motion of a spin-$\frac{1}{2}$ particle in $\mathbb{R}^3$ and occurs in various areas of modern physics and mathematics. In 1973, Ablowitz, Kaup, Newell and Segur [1] made a significant contribution for the solution of the initial-value problem for a broad class of nonlinear evolution equations, which increased interest in direct and inverse problems for Dirac operators in both physics and mathematics. Thaller presented mathematical and physical aspects of Dirac equation in the book [20] and we refer the reader to the bibliography quoted therein.

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The inverse problem in scattering theory deals with the reconstruction of potential from knowing scattering data. In 1966, Gasymov and Levitan investigated determination of the Dirac system from the scattering phase in [11] and the inverse problem for the Dirac system in [10]. In 1968, Gasymov completely solved an inverse problem of scattering theory for a system of Dirac equations of order $2n$ in the manuscript [9]. The inverse scattering theory for Dirac operators without spectral parameter in boundary condition has been developed by many authors in [2], [5], [6], [12], [15], [18], [19]. Problems with boundary conditions depending on spectral parameter have been studied in [3], [4], [7], [8], [13], [14], [16], [17] and other papers.

In present work we consider the boundary value problem generated by the canonical system of Dirac differential equations

$$By' + mTy + \Omega(x)y = \lambda y \tag{1}$$

on the half line $[0, \infty)$ with the boundary condition

$$(\alpha_0 y_1(0) - \beta_0 y_2(0)) + (\alpha_1 y_1(0) - \beta_1 y_2(0))\lambda + (\alpha_2 y_1(0) -$$

$$-\beta_2 y_2(0))\lambda^2 + (\alpha_3 y_1(0) - \beta_3 y_2(0))\lambda^3 = 0, \tag{2}$$

where λ is a spectral parameter, $m > 0$ is the mass and

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Omega(x) = \begin{pmatrix} p\,(x) & q\,(x) \\ q\,(x) & -p\,(x) \end{pmatrix}, \quad y = \begin{pmatrix} y_1\,(x) \\ y_2\,(x) \end{pmatrix}.$$

Here, $\Omega\,(x)$ is a canonical potential matrix function, where the components $p\,(x)$ and $q\,(x)$ are real valued functions and hold the following estimates for positive numbers c and ϵ

$$|p\,(x)| \leq \frac{c}{(1+x)^{2+\epsilon}}, \qquad |q\,(x)| \leq \frac{c}{(1+x)^{1+\epsilon}}. \tag{3}$$

Assume that the following relations hold for $\alpha_i,\ \beta_i \in \mathbb{R}\ \ (i = \overline{0,3})$, $\alpha_3 \neq 0$, $\beta_3 \neq 0$,

$$\delta_{ik} \geq 0,\ k = 1;\ \ \delta_{ik} = 0,\ k = 2, 3, \ \text{ where } \delta_{ik} = \alpha_{i+k}\beta_i - \alpha_i\beta_{i+k}. \tag{4}$$

We study the inverse scattering problem for a system for Dirac equations with spectral parameter in boundary condition. In progress, we will extend the Gasymov method to the boundary value problem (1)-(3), which also introduces a process for inverse problems where boundary condition depends on the spectral parameter nonlinearly.

The remaining paper is organized as follows. In Section 2, we obtain the quantities $S\,(\lambda)\,;\ \lambda_1, \ldots \lambda_n;\ m_1, \ldots m_n$ to be scattering data corresponding to the boundary value problem (1)-(3) and investigate their properties. In Section 3, the main equation is derived in order to study inverse scattering problem. Finally, we show that the main equation constructed only on the basis of the scattering data has a unique solution and the reconstruction of the potential matrix function of equation (1) is presented in Section 4.

2. Scattering Data

In this section, we establish the scattering function and the spectrum for the boundary value problem (1)-(3).

Let $\Omega(x) = 0$, then the vector function $f_0(x, \lambda) = \begin{pmatrix} \frac{\lambda+m}{k} \\ -i \end{pmatrix} e^{ikx}$ is a solution of (1), where $k = \lambda\sqrt{1 - \frac{m^2}{\lambda^2}}$, $|\lambda| > m$.

Assume that the condition (3) holds. Then, as known in [?], there exists a unique vector solution $f(x, \lambda) = \begin{pmatrix} f_1(x, \lambda) \\ f_2(x, \lambda) \end{pmatrix}$ which tends to $f_0(x, \lambda)$ as $x \to \infty$, $Im\lambda \geq 0$ and can be expressed as

$$f(x, \lambda) = f_0(x, \lambda) + \int_x^\infty A(x, t) f_0(t, \lambda)\, dt, \tag{5}$$

where the components of the matrix kernel $A(x, t)$ holds the following estimates

$$|A_{ij}| \leq \frac{c_1}{(1 + x)(1 + t)^{1+\epsilon}}, \quad i \neq j,$$

$$|A_{ii}| \leq \frac{c_2}{(1 + t)^{1+\epsilon}}, \quad i = 1, 2,$$

and the matrix kernel $A(x, t)$ possesses the following relation

$$BA(x, x) - A(x, x) B = \Omega(x).$$

Let $W[y(x, \lambda), z(x, \lambda)] = \widetilde{y}(x, \lambda)Bz(x, \lambda) = y_1z_2 - y_2z_1$ denote the Wronskian of the vector functions $y(x, \lambda)$ and $z(x, \lambda)$, where $\widetilde{y}$ is the transposed matrix of y. For all λ in the intervals $(-\infty, -m)$ and (m, ∞), $f(x, \lambda)$ and $\overline{f(x, \lambda)}$ constitute a fundamental system of solutions of equation (1) and their Wronskian is independent of x and and equal to $2i\frac{\lambda+m}{k}$.

Let us use the following notations:

$$a(\lambda) := \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \alpha_3\lambda^3, \quad b(\lambda) := \beta_0 + \beta_1\lambda + \beta_2\lambda^2 + \beta_3\lambda^3$$

and denote $\psi(x, \lambda) = \begin{pmatrix} \psi_1(x, \lambda) \\ \psi_2(x, \lambda) \end{pmatrix}$ by the solution of (1) satisfying the conditions

$$\psi_1(0, \lambda) = b(\lambda), \quad \psi_2(0, \lambda) = a(\lambda).$$

Obviously, this solution holds the boundary condition (2).

After stating the above preliminaries, we can now present the following lemmas.

Lemma 1. *The identity*

$$2i\frac{\lambda + m}{k}\frac{\psi(x, \lambda)}{\Phi(\lambda)} = \overline{f(x, \lambda)} - S(\lambda) f(x, \lambda) \tag{6}$$

holds for all λ *in the intervals* $(-\infty, -m)$ *and* (m, ∞)*, where*

$$S(\lambda) = \overline{\Phi(\lambda)}[\Phi(\lambda)]^{-1} \tag{7}$$

and

$$S^{-1}(\lambda) = \overline{S(\lambda)}, \quad |S(\lambda)| = 1.$$

Proof. Since $\overline{f(x,\lambda)}$ and $f(x,\lambda)$ for real λ and $|\lambda| > m$ constitute a linearly independent matrix system of solutions of (1), then any solution of (1) can be be expressed as

$$\psi(x,\lambda) = c_1(\lambda) f(x,\lambda) + c_2(\lambda)\overline{f(x,\lambda)}.$$

If we evaluate the following Wronskians of $f(x,\lambda)$ and $\psi(x,\lambda)$:

$$\widetilde{f}(0,\lambda) B\psi(0,\lambda) = c_2(\lambda) 2i\frac{\lambda+m}{k},$$
$$f^*(0,\lambda) B\psi(0,\lambda) = -c_1(\lambda) 2i\frac{\lambda+m}{k}$$

and take $\Phi(\lambda) = \widetilde{f}(0,\lambda)B\psi(0,\lambda)$, then we find $c_1(\lambda)$ and $c_2(\lambda)$ and hence $\psi(x,\lambda)$ as

$$\psi(x,\lambda) = \frac{k}{2i(\lambda+m)}\left[-\overline{\Phi(\lambda)} f(x,\lambda) + \Phi(\lambda)\overline{f(x,\lambda)}\right]. \tag{8}$$

To show $\Phi(\lambda) \neq 0$ for all real λ, $|\lambda| > m$, we assume the contrary. Then there exists λ_0, $|\lambda_0| > m$ such that

$$\widetilde{f}(0,\lambda_0) B\psi(0,\lambda_0) = a(\lambda_0) f_1(0,\lambda_0) - b(\lambda_0) f_2(0,\lambda_0) = 0.$$

Since $a(\lambda) = \overline{a(\lambda)}$ and $b(\lambda) = \overline{b(\lambda)}$ for all real λ, $|\lambda| > m$, we get

$$2i\frac{\lambda_0+m}{k} = W[f(0,\lambda_0), \overline{f(0,\lambda_0)}] = \left(\frac{b(\lambda_0)}{a(\lambda_0)} - \frac{\overline{b(\lambda_0)}}{\overline{a(\lambda_0)}}\right)|f_2(0,\lambda_0)|^2 = 0,$$

which yields $\lambda_0 = -m$, hence we arrive at a contradiction. This proves the claim.

Dividing equality (8) by $\frac{k}{2i(\lambda+m)}\Phi(\lambda)$, the identity (6) is obtained and $S(\lambda)$ is defined with (7).

From the definition of $S(\lambda)$, we obtain

$$S(\lambda) = \frac{\overline{\Phi(\lambda)}}{\Phi(\lambda)} = \overline{\left[\frac{\Phi(\lambda)}{\overline{\Phi(\lambda)}}\right]} = \left[\overline{S(\lambda)}\right]^{-1}$$

and

$$|S(\lambda)| = \left|\frac{\overline{\Phi(\lambda)}}{\Phi(\lambda)}\right| = 1.$$

Thus, the lemma is proved. ◀

The function $S(\lambda)$ is called the *scattering function* of the boundary value problem (1)-(3).

From the definition of $S(\lambda)$, we obtain that the function $S_0(\lambda) - S(\lambda)$ is of an integrable square over $(-\infty, -m]$ and $[m, \infty)$ where $S_0(\lambda) = \frac{\alpha_3 - i\beta_3}{\alpha_3 + i\beta_3}$. In fact, using (5) and substituting related expressions into the function $\Phi(\lambda)$, we find

$$\Phi(\lambda) = \lambda^3[\alpha_3 + i\beta_3 + O\left(\frac{1}{\lambda}\right)].$$

Taking this into account, we get

$$S(\lambda) = \frac{\alpha_3 - i\beta_3 + O\left(\frac{1}{\lambda}\right)}{\alpha_3 + i\beta_3 + O\left(\frac{1}{\lambda}\right)} = \frac{\alpha_3 - i\beta_3}{\alpha_3 + i\beta_3} + O\left(\frac{1}{\lambda}\right), \quad |\lambda| \to \infty.$$

Let denote $S_0(\lambda) := \frac{\alpha_3 - i\beta_3}{\alpha_3 + i\beta_3}$. Hence we have $S_0(\lambda) - S(\lambda) = O\left(\frac{1}{\lambda}\right)$ as $|\lambda| \to \infty$, which shows the claim.

Next, let us investigate properties of $\Phi(\lambda)$ in the upper half plane.

Lemma 2. *The function $\Phi(\lambda)$ is analytic in the upper half plane* $\operatorname{Im}\lambda > 0$*, continuous along the real axis except at $\lambda = m$ and has only a finite number of zeros in the interval $(-m, m)$. All the zeros in $(-m, m)$ are simple.*

Proof. The functions $f_1(0, \lambda)$ and $f_2(0, \lambda)$ are continuous for all real $\lambda \neq m$ and analytic in the upper plane ($\operatorname{Im}\lambda > 0$). Then, it follows that the function $\Phi(\lambda)$ has the same properties.

Let μ ($Im\mu > 0$ or $\mu \in (-m, m)$) be a zero of the function $\Phi(\lambda)$. It satisfies the equation (1)

$$Bf'(x, \mu) + mTf(x, \mu) + \Omega(x) f(x, \mu) = \mu f(x, \mu). \tag{9}$$

By going over to the Hermitian conjugates of the equation, we obtain the following equation

$$-f^{*\prime}(x, \mu) B + mf^*(x, \mu) T + f^*(x, \mu) \Omega(x) = \overline{\mu} f^*(x, \mu), \tag{10}$$

where the function $f^*(x, \mu)$ denotes the transposed vector function $\overline{f(x, \mu)}$. Multiplying (9) on the left by $f^*(x, \mu)$ and (10) on the right by $f(x, \mu)$, subtracting the second from the first, and finally integrating the result with respect to x over $(0, \infty)$, we get

$$f^*(0, \lambda) B f(0, \lambda) + (\mu - \overline{\mu}) \int_0^\infty f^*(x, \mu) f(x, \mu) dx = 0. \tag{11}$$

If μ is a zero of $\Phi(\lambda)$, then we have $f_1(0, \mu) = \frac{b(\mu)}{a(\mu)} f_2(0, \mu)$. Hence

$$f^*(0, \lambda) B f(0, \lambda) = \left(\frac{\overline{b(\lambda_0)}}{\overline{a(\lambda_0)}} - \frac{b(\lambda_0)}{a(\lambda_0)} \right) |f_2(0, \lambda_0)|^2 =$$

$$= \frac{|f_2(0,\mu)|^2}{|a(\mu)|^2}\left[(\mu-\overline{\mu})\sum_{i=0}^{2}(\alpha_{i+1}\beta_i-\alpha_i\beta_{i+1})|\mu|^{2i}+\right.$$

$$\left.+(\mu^2-\overline{\mu}^2)\sum_{i=0}^{1}(\alpha_{i+2}\beta_i-\alpha_i\beta_{i+2})|\mu|^{2i}+(\mu^3-\overline{\mu}^3)(\alpha_3\beta_0-\alpha_0\beta_3)\right].$$

If we substitute the result in (11) and take the condition (4) into account, we obtain

$$(\mu-\overline{\mu})\left[\frac{|f_2(0,\mu)|^2}{|a(\mu)|^2}\sum_{i=0}^{2}(\alpha_{i+1}\beta_i-\alpha_i\beta_{i+1})|\mu|^{2i}+\int_0^\infty f^*(x,\mu)f(x,\mu)dx\right]=0. \tag{12}$$

The condition (4) leads that the expression in the parentheses is positive, which implies $\mu-\overline{\mu}=0$, i.e. μ is real. Hence we conclude that $\mu\in(-m,m)$.

Let us prove that there are only finitely many zeros. We assume that δ denotes the infimum of the distances between two neighboring zeros of $\Phi(\lambda)$, and show $\delta>0$. Suppose the contrary and let $\{\lambda_k\}$ and $\left\{\widetilde{\lambda_k}\right\}$ be two sequences of zeros of the function $\Phi(\lambda)$ such that

$$\lim_{k\to\infty}\left(\widetilde{\lambda_k}-\lambda_k\right)=0, \quad -m\le\lambda_k<\widetilde{\lambda_k}<0.$$

For A large enough, the estimates

$$if_1(x,\lambda)>\frac{1}{2}\frac{\lambda+m}{\sqrt{m^2-\lambda^2}}e^{-\sqrt{m^2-\lambda^2}x}, \quad if_2(x,\lambda)>\frac{1}{2}e^{-\sqrt{m^2-\lambda^2}x}$$

hold uniformly with respect to $x\in[A,\infty)$ and $\lambda\in(-m,m)$, which yields

$$\int_A^\infty f^*(x,\lambda_k)\,f\left(x,\widetilde{\lambda_k}\right)dx>\frac{1}{4}\frac{e^{-\left(\sqrt{m^2-\lambda_k^2}+\sqrt{m^2-\widetilde{\lambda_k}^2}\right)A}}{\sqrt{m^2-\lambda_k^2}+\sqrt{m^2-\widetilde{\lambda_k}^2}}.$$

Letting $k\to\infty$, it follows that

$$\lim_{k\to\infty}\int_A^\infty f^*(x,\lambda_k)\,f\left(x,\widetilde{\lambda_k}\right)dx=+\infty. \tag{13}$$

On the other hand, the equality (12) yields

$$0=\int_0^\infty f^*(x,\lambda_k)\,f\left(x,\widetilde{\lambda_k}\right)dx+\frac{\overline{f_2(0,\lambda_k)}f_2\left(0,\widetilde{\lambda_k}\right)}{a\left(\widetilde{\lambda_k}\right)a(\lambda_k)}\sum_{i=0}^{2}(\alpha_{i+1}\beta_i-\alpha_i\beta_{i+1})\lambda_k^i\widetilde{\lambda_k}^i=$$

$$=\int_0^A f^*(x,\lambda_k)\left[f\left(x,\widetilde{\lambda_k}\right)-f(x,\lambda_k)\right]dx+\int_0^A f^*(x,\lambda_k)\,f(x,\lambda_k)\,dx+$$

$$+\int_A^\infty f^*(x,\lambda_k)\, f\left(x,\widetilde{\lambda_k}\right)dx+\frac{\overline{f_2(0,\lambda_k)}f_2\left(0,\widetilde{\lambda_k}\right)}{a\left(\widetilde{\lambda_k}\right)a(\lambda_k)}\sum_{i=0}^{2}(\alpha_{i+1}\beta_i-\alpha_i\beta_{i+1})\lambda_k^i\widetilde{\lambda_k}^i,$$

and letting $k\to\infty$, we get

$$\lim_{k\to\infty}\int_A^\infty f^*(x,\lambda_k)\, f\left(x,\widetilde{\lambda_k}\right)dx\le 0. \tag{14}$$

Comparing (13) and (14), we arrive at the contradiction. We conclude that $\delta>0$ and hence the function $\Phi(\lambda)$ has only a finite number of zeros.

Finally, let us show that all zeros of the function $\Phi(\lambda)$ in $(-m,m)$ are simple. When we differentiate the identity

$$Bf'(x,\lambda)+mTf(x,\lambda)+\Omega(x)f(x,\lambda)=\lambda f(x,\lambda) \tag{15}$$

with respect to λ, assuming that λ lies in $(-m,m)$, and go over to the hermitian conjugates of the matrices, we get

$$-\left[\dot f^*(x,\lambda)\right]'B+m\dot f^*(x,\lambda)T+\dot f^*(x,\lambda)\Omega(x)=\lambda\dot f^*(x,\lambda)+f^*(x,\lambda), \tag{16}$$

where $\dot f$ denotes differentiation with respect to λ. Multiplying (15) on the left by $\dot f^*(x,\lambda)$ and (16) on the right by $f(x,\lambda)$ and subtracting the second from the first, we obtain

$$\dot f^*(x,\lambda)Bf'(x,\lambda)+\left[\dot f^*(x,\lambda)\right]'Bf(x,\lambda)=-f^*(x,\lambda)f(x,\lambda).$$

Integrating the result with respect to x over $(0,\infty)$, it follows that

$$\dot f^*(0,\lambda)Bf(0,\lambda)=\int_0^\infty f^*(x,\lambda)f(x,\lambda)\,dx.$$

Let λ_j be a zero of the function $\Phi(\lambda)$. The functions $\Phi(\lambda)$ and $\dot\Phi(\lambda)$ yields that

$$\dot f^*(0,\lambda_j)Bf(0,\lambda_j)=-\frac{\dot\Phi(\lambda_j)f_2(0,\lambda_j)}{a(\lambda_j)}+\left[\dot a(\lambda_j)b(\lambda_j)-a(\lambda_j)\dot b(\lambda_j)\right]\left[\frac{f_2(0,\lambda_j)}{a(\lambda_j)}\right]^2$$

and hence

$$-\frac{\dot\Phi(\lambda_j)f_2(0,\lambda_j)}{a(\lambda_j)}=$$

$$=-\left[\frac{f_2(0,\lambda_j)}{a(\lambda_j)}\right]^2\sum_{i=0}^{2}(\alpha_{i+1}\beta_i-\alpha_i\beta_{i+1})\lambda_j^{2i}+\int_0^\infty f^*(x,\lambda)f(x,\lambda)\,dx. \tag{17}$$

Since $f_2(0,\lambda_j)$ is pure imaginary and the condition (4) holds, the right side of the equality is positive. This shows $\dot\Phi(\lambda_j)\neq 0$, i.e., the zeros of $\Phi(\lambda)$ are simple. The lemma is proved. ◄

The numbers $m_j, j = 1, \ldots n$, are defined with

$$m_j^{-2} \equiv \int_0^\infty f^* (x, \lambda_j) f (x, \lambda_j) dx + \frac{|f_2 (0, \lambda_j)|^2}{|a(\lambda_j)|^2} \sum_{i=0}^{2} (\alpha_{i+1}\beta_i - \alpha_i\beta_{i+1}) \lambda_j^{2i}$$

and called *norming numbers* for boundary value problem (1)-(3).

Consequently, in this section we have defined the scattering function $S(\lambda)$ for boundary value problem (1)-(3) and have shown that the denominator of scattering function has only finitely many discrete zeros $\lambda_1, ..., \lambda_n$ in the interval $(-m, m)$ with the norming numbers $m_1, ..., m_n$. Now, we can give the following definition.

Definition. *The set of values* $\left\{S(\lambda), \lambda_k, m_k;\ k = \overline{1, n}\right\}$ *is called the scattering data for boundary value problem (1)-(3).*

3. The Main Equation

In this section, we present the main equation in order to discuss of inverse scattering problem.

Theorem 1. *For every fixed* $x \geq 0$*, the kernel* $A(x, t)$ *of the solution (5) satisfies the integral equation which is called the main equation*

$$F(x+y) + A(x, y) + \int_x^\infty A(x, t)F(t + y)dt = 0, \quad y > x, \tag{18}$$

where

$$F(x+y) = F_s(x+y) - \sum_{j=1}^{n} 2m_j^2 f_0(x, \lambda_j) \widetilde{f_0}(y, \lambda_j) \tag{19}$$

and

$$F_s(x) = \operatorname{Re} \frac{1}{2\pi} \int_{|\lambda|>m} (S_0(\lambda) - S(\lambda)) \begin{pmatrix} \frac{\lambda+m}{k} & -i \\ -i & -\frac{k}{\lambda+m} \end{pmatrix} e^{ikx} d\lambda. \tag{20}$$

Proof. Substituting (5) into the identity (6), we obtain

$$2i\frac{\lambda+m}{k}\frac{\psi(x,\lambda)}{\Phi(\lambda)} - \begin{pmatrix} \frac{\lambda+m}{k} \\ i \end{pmatrix} e^{-ikx} + S_0(\lambda) \begin{pmatrix} \frac{\lambda+m}{k} \\ -i \end{pmatrix} e^{ikx} =$$

$$= \int_x^\infty A(x,t) \begin{pmatrix} \frac{\lambda+m}{k} \\ i \end{pmatrix} e^{-ikt} dt - S_0(\lambda) \int_x^\infty A(x,t) \begin{pmatrix} \frac{\lambda+m}{k} \\ -i \end{pmatrix} e^{ikt} dt +$$

$$+ (S_0(\lambda) - S(\lambda)) \begin{pmatrix} \frac{\lambda+m}{k} \\ -i \end{pmatrix} e^{ikx} + (S_0(\lambda) - S(\lambda)) \int_x^\infty A(x,t) \begin{pmatrix} \frac{\lambda+m}{k} \\ -i \end{pmatrix} e^{ikt} dt.$$

Multiplying both sides of this equality by $\frac{1}{2\pi}\frac{k}{\lambda+m}\left(\frac{\lambda+m}{k},-i\right)e^{iky}$ and integrating with respect to λ over $(-\infty,-m)$ and (m,∞), we get

$$\operatorname{Re}\frac{1}{2\pi}\int\limits_{|\lambda|>m}2i\frac{\psi(x,\lambda)}{\varPhi(\lambda)}\left(\frac{\lambda+m}{k},-i\right)e^{iky}d\lambda+$$

$$+\operatorname{Re}\frac{1}{2\pi}\int\limits_{|\lambda|>m}S_0(\lambda)\begin{pmatrix}\frac{\lambda+m}{k} & -i\\ -i & -\frac{k}{\lambda+m}\end{pmatrix}e^{ik(x+y)}d\lambda-$$

$$-\operatorname{Re}\frac{1}{2\pi}\int\limits_{|\lambda|>m}\begin{pmatrix}\frac{\lambda+m}{k} & -i\\ i & \frac{k}{\lambda+m}\end{pmatrix}e^{-ik(x-y)}d\lambda=$$

$$=\operatorname{Re}\frac{1}{2\pi}\int\limits_{|\lambda|>m}\int\limits_x^\infty A(x,t)\begin{pmatrix}\frac{\lambda+m}{k} & -i\\ i & \frac{k}{\lambda+m}\end{pmatrix}e^{-ik(t-y)}dtd\lambda-$$

$$-\operatorname{Re}\frac{1}{2\pi}\int\limits_{|\lambda|>m}S_0(\lambda)\int\limits_x^\infty A(x,t)\begin{pmatrix}\frac{\lambda+m}{k} & -i\\ -i & -\frac{k}{\lambda+m}\end{pmatrix}e^{ik(t+y)}dtd\lambda+$$

$$+\operatorname{Re}\frac{1}{2\pi}\int\limits_{|\lambda|>m}(S_0(\lambda)-S(\lambda))\begin{pmatrix}\frac{\lambda+m}{k} & -i\\ -i & -\frac{k}{\lambda+m}\end{pmatrix}e^{ik(x+y)}d\lambda+$$

$$+\operatorname{Re}\frac{1}{2\pi}\int\limits_{|\lambda|>m}(S_0(\lambda)-S(\lambda))\int\limits_x^\infty A(x,t)\begin{pmatrix}\frac{\lambda+m}{k} & -i\\ -i & -\frac{k}{\lambda+m}\end{pmatrix}e^{ik(t+y)}dtd\lambda. \tag{21}$$

For the first term on the left, we evaluate

$$Re\frac{1}{2\pi}\int\limits_{|\lambda|>m}\begin{pmatrix}\frac{\lambda+m}{k} & -i\\ i & \frac{k}{\lambda+m}\end{pmatrix}e^{-ik(t-y)}d\lambda=\delta(t-y)\begin{pmatrix}1\,0\\0\,1\end{pmatrix}=\delta(t-y)I_2,$$

where $\delta(x)$ is a Dirac delta function, and using the result we obtain

$$\operatorname{Re}\frac{1}{2\pi}\int\limits_{|\lambda|>m}\int\limits_x^\infty A(x,t)\begin{pmatrix}\frac{\lambda+m}{k} & -i\\ i & \frac{k}{\lambda+m}\end{pmatrix}e^{-ik(t-y)}dtd\lambda=$$

$$=\int\limits_x^\infty A(x,t)\,Re\frac{1}{2\pi}\int\limits_{|\lambda|>m}\begin{pmatrix}\frac{\lambda+m}{k} & -i\\ i & \frac{k}{\lambda+m}\end{pmatrix}e^{-ik(t-y)}d\lambda dt=$$

$$=\int\limits_x^\infty A(x,t)\,\delta(t-y)dt=A(x,y).$$

For the second term on the left, using the fact $A(x,-y)=0$ for $y>x$, we obtain

$$\operatorname{Re}\frac{1}{2\pi}\int\limits_{|\lambda|>m} S_0(\lambda)\int\limits_x^{\infty} A(x,t)\begin{pmatrix}\frac{\lambda+m}{k} & -i\\ -i & -\frac{k}{\lambda+m}\end{pmatrix} e^{ik(t+y)}dtd\lambda=$$

$$=\int\limits_x^{\infty} A(x,t)\operatorname{Re}\frac{1}{2\pi}\int\limits_{|\lambda|>m}\frac{\alpha_3-i\beta_3}{\alpha_3+i\beta_3}\begin{pmatrix}\frac{\lambda+m}{k} & -i\\ -i & -\frac{k}{\lambda+m}\end{pmatrix} e^{ik(t+y)}dtd\lambda=$$

$$=\frac{\alpha_3^2-\beta_3^2}{\alpha_3^2+\beta_3^2}\int\limits_x^{\infty} A(x,t)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\delta(-t-y)dt=0.$$

Let us define $F_s(x)$ with (20). Therefore, for $y>x$, the right side of (21) equals

$$F_s(x+y)+A(x,y)+\int\limits_x^{\infty} A(x+t)F_s(t+y)dt.$$

On the other side, using the residue theorem and the formula (17) we have

$$\operatorname{Re}\frac{1}{2\pi}\int\limits_{|\lambda|>m} 2i\frac{\psi(x,\lambda)}{\Phi(\lambda)}\left(\frac{\lambda+m}{k},-i\right)e^{iky}d\lambda-\operatorname{Re}\frac{1}{2\pi}\int\limits_{|\lambda|>m}\begin{pmatrix}\frac{\lambda+m}{k} & -i\\ i & \frac{k}{\lambda+m}\end{pmatrix} e^{-ik(x-y)}d\lambda+$$

$$+\operatorname{Re}\frac{1}{2\pi}\int\limits_{|\lambda|>m} S_0(\lambda)\begin{pmatrix}\frac{\lambda+m}{k} & -i\\ -i & -\frac{k}{\lambda+m}\end{pmatrix} e^{ik(x+y)}d\lambda=$$

$$=-\sum_{j=1}^{n} Res\left[\frac{2\psi(x,\lambda)}{\Phi(\lambda)}\left(\frac{\lambda+m}{k},-i\right)e^{iky}\right]\Bigg|_{\lambda=\lambda_j}=$$

$$=-\sum_{j=1}^{n}\frac{2a(\lambda_j)f(x,\lambda_j)}{\dot{\Phi}(\lambda_j)\,f_2(0,\lambda_j)}\left(\frac{\lambda_j+m}{k},-i\right)e^{iky}=\sum_{j=1}^{n}2m_j^2 f(x,\lambda_j)\left(\frac{\lambda_j+m}{k},-i\right)e^{iky}=$$

$$=\sum_{j=1}^{n}2m_j^2\left[\begin{pmatrix}\frac{\lambda_j+m}{k}\\ -i\end{pmatrix}e^{ikx}+\int\limits_x^{\infty}A(x,t)\begin{pmatrix}\frac{\lambda_j+m}{k}\\ -i\end{pmatrix}e^{ikt}dt\right]\left(\frac{\lambda+m}{k},-i\right)e^{iky}.$$

Substituting this value into the left side of (21), we get the integral equation (18), where $F(x)$ is defined with the formula (19). This completes the proof of theorem. ◀

4. Solvability of the Main Equation

The inverse scattering problem consists in recovering the coefficient $\Omega(x)$ from the scattering data of the boundary value problem (1)-(3). In this section, we examine the solvability of the main equation and present that the potential matrix function $\Omega(x)$ can be recovered uniquely from the scattering data.

Theorem 2. *For every fixed x, the main equation (20) has a unique matrix solution with components in $L_2(x,\infty)$.*

Proof. In order to determine $\Omega(x)$, it is sufficient to know $A(x,t)$. Therefore, assume that the data $\left\{S(\lambda), \lambda_k, m_k;\ k=\overline{1,n}\right\}$ are given. Then we write the matrix function $F(x)$ by the formula (19) and construct the equation (18). Let us take the kernel $A(x,t)$ of the solution (5) as unknown and regard this equation as a Fredholm-type matrix equation in the space of matrix functions with components in $L_2(x,\infty)$ for every fixed x. Denote $f(t)=A(x,t)$, then we get the corresponding homogeneous equation for each fixed $x\geq 0$

$$f(y)+\int_x^\infty f(t)F(t+y)dt=0. \tag{22}$$

The transition function $F(x)$ possesses similar properties to the transition function of the problem without the spectral parameter in the boundary condition ([9]). With the help of the proof of Theorem 2.3.1 in [9], the result is easily obtained that the equation (22) has only the zero solution with components in $L_2(x,\infty)$, which proves the claim and the theorem is proved. ◄

Corollary. *The scattering data of the boundary value problem (1)-(3) determine the potential matrix function $\Omega(x)$ in equation (1) uniquely.*

Proof. The main equation (18) is constructed only on the basis of the scattering data, and by Theorem 2, it has a unique solution $A(x,y)$ for every $x\geq 0$. Hence, we have the matrix kernel $A(x,y)$ of the solution (5), and obtain the potential matrix function as

$$\Omega(x)=BA(x,x)-A(x,x)B.$$

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