Baku Mathematical Journal
2024, Vol. 3, No 2, PP. 1-10
https://doi.org/10.32010/j.bmj.202X.XX

THE A-INTEGRAL AND CALDERON-ZYGMUND OPERATORS

A.F. HUSEYNLI

Received: date / Revised: date / Accepted: date

In memory of M. G. Gasymov on his 85th birthday

Abstract. *It is known that the function obtained by the action of the Calderon-Zygmund operator on a Lebesgue integrable function can be non-Lebesgue integrable. In this paper, we prove that the function obtained by the action of the Calderon-Zygmund operator on a Lebesgue integrable function is A-integrable and derive an analogue of the Riesz equality.*

Keywords: singular integral, Calderon-Zygmund operator, A-integral

Mathematics Subject Classification (2020): 26A39, 42B20

1. Introduction

A Calderon-Zygmund operator is a principal value convolution operator

$$(Tf)(x) = \lim_{\varepsilon\to 0+} \int_{\{y\in R^d\,:\,|x-y|>\varepsilon\}} K(x-y)f(y)dy,$$

where dy denotes Lebesgue measure in R^d and

$$K(x) = \frac{\Omega(x)}{|x|^d}, \quad x \neq 0,$$

$\Omega(x)$ is a homogeneous function of degree 0, continuously by Hölder with exponent $\alpha \in (0,1]$ on the unit sphere S^{d-1} and with zero integral on the S^{d-1}.

Let $D \subset R^d$ be a bounded domain and $f \in L_1(D)$. In the present paper we consider the corresponding modification of T. Namely, the restricted Calderon-Zygmund operator T_D is defined as

$$(T_D f)(x) = T(\chi_D f)(x) = \lim_{\varepsilon\to 0+} \int_{\{y\in D\,:\,|x-y|>\varepsilon\}} K(x-y)f(y)dy, \qquad x \in D.$$

Aynur F. Huseynli
Baku State University, Baku, Azerbaijan
E-mail: 1919-bdu@mail.ru

From the theory of singular integrals (see [14]) it is known that the Calderon-Zygmund operator is a bounded operator in the space $L_p(D)$, $1 < p < \infty$, that is, if $f \in L_p(D)$, then $T_D(f) \in L_p(D)$ and

$$\|T_D f\|_{L_p} \leq C_p \|f\|_{L_p}. \tag{1}$$

In the case $f \in L_1(D)$ only the weak inequality holds:

$$m\{x \in D: \ |(T_D f)(x)| > \lambda\} \leq \frac{C_1}{\lambda} \|f\|_{L_1}, \tag{2}$$

where m stands for the Lebesgue measure, C_p, C_1 are constants independent of f. In the case $f \in L_1(D)$ from inequalities (1), (2) it follows that

$$m\{x \in D: \ |(T_D f)(x)| > \lambda\} = o(1/\lambda), \quad \lambda \to +\infty. \tag{3}$$

Indeed, if $f \in L_1(D)$, then for every $\varepsilon > 0$ there exists $n \in N$ such that $\|f - [f]^n\|_{L_1} \leq \frac{\varepsilon}{4C_1}$, where $[f(x)]^n = f(x)$ for $|f(x)| \leq n$ and $[f(x)]^n = 0$ for $|f(x)| > n$. It follows from (2) that

$$m\{x \in D: \ |T_D(f - [f]^n)(x)| > \lambda/2\} \leq \frac{2C_1}{\lambda} \cdot \|f - [f]^n\|_{L_1} \leq \frac{\varepsilon}{2\lambda}. \tag{4}$$

Since the function $[f(x)]^n$ is bounded, $[f]^n \in L_p(D)$ for every $p \geq 1$; whence $T_D([f]^n) \in L_p(D)$ for every $p > 1$. Therefore $T_D([f]^n) \in L_1(D)$. It follows that for sufficiently large values of $\lambda > 0$

$$\frac{\lambda}{2} m\{x \in D: \ |(T_D[f]^n)(x)| > \lambda/2\} \leq \int_{\{x \in D: \ |(T_D[f]^n)(x)| > \frac{\lambda}{2}\}} |(T_D[f]^n)(x)| dx < \frac{\varepsilon}{4}. \tag{5}$$

We obtain from (4) and (5) that for sufficiently large values of $\lambda > 0$

$$m\{x \in D: \ |(T_D f)(x)| > \lambda\}$$

$$\leq m\{x \in D: \ |T_D(f - [f]^n)(x)| > \lambda/2\} + m\{x \in D: \ |(T_D[f]^n)(x)| > \lambda/2\} < \frac{\varepsilon}{\lambda}.$$

This means that the condition (3) holds.

Note that in the case of $f \in L_1(D)$ the function $T_D f$ is not Lebesgue integrable on D. In the present paper, we prove that if $f \in L_1(D)$, then the function $T_D f$ is A-integrable on D and derive an analogue of the Riesz equality.

2. A-integral

For a measurable complex function $f(x)$ on domain D we set

$$[f(x)]_n = [f(x)]^n = f(x) \text{ for } |f(x)| \le n,$$

$$[f(x)]_n = n \cdot \operatorname{sgn} f(x),\ [f(x)]^n = 0 \text{ for } |f(x)| > n,\ n \in N,$$

where $\operatorname{sgn} w = \frac{w}{|w|}$ for $w \neq 0$ and $\operatorname{sgn} 0 = 0$.

In 1928, Titchmarsh [15] introduced the notions of Q- and Q'-integrals of a function measurable on D.

Definition 1. *If the finite limit $\lim\limits_{n\to\infty} \int_D [f(x)]_n dx$ ($\lim\limits_{n\to\infty} \int_D [f(x)]^n dx$, respectively) exists, then f is said to be Q-integrable (Q'-integrable, respectively) on D; that is, $f \in Q(D)$ ($f \in Q'(D)$). The value of this limit is referred to as the Q-integral (Q'-integral) of this function and is denoted by $(Q)\int_D f(x)dx$ ($(Q')\int_D f(x)dx$.*

In the same paper, Titchmarsh when studying properties of trigonometric series conjugate to Fourier series of Lebesgue integrable functions, established that the Q-integrability leads to a series of natural results. A very uncomfortable fact impeding the application of Q-integrals and Q'-integrals when dealing with diverse problems of function theory is the absence of the additivity property; that is, the Q-integrability (Q'-integrability) of two functions does not imply the Q-integrability (Q'-integrability) of their sum. If one adds the condition

$$m\{x \in D : |f(x)| > \lambda\} = o(1/\lambda), \quad \lambda \to +\infty \tag{6}$$

to the definition of Q-integrability (Q'-integrability) of a function f, then the Q-integral and Q'-integral coincide ($Q(D) = Q'(D)$), and these integrals become additive.

Definition 2. *If $f \in Q'(D)$ (or $f \in Q(D)$) and condition (6) holds, then f is said to be A-integrable on D, $f \in A(D)$, and the limit $\lim\limits_{n\to\infty} \int_D [f(x)]_n dx$ (or the limit $\lim\limits_{n\to\infty} \int_D [f(x)]^n dx$) is denoted in this case by $(A)\int_D f(x)dx$.*

The properties of Q- and Q'-integrals were investigated in [2]-[5], [9], [10]; for the applications of A-, Q- and Q'-integrals in the theory of functions of real and complex variables we refer the reader to [1], [6], [7], [12], [13], [16], [17].

3. A-integrability and Riesz's Equation for Calderon-Zygmund Operators

From the properties of singular integrals it follows that (see [8]) if $f \in L_p(D)$, $p > 1$ and $g \in L_q(D)$, $q > 1$, $1/p + 1/q = 1$, then

$$\int_D g(x)(T_D f)(x)dx = \lim_{\varepsilon\to 0+} \iint_{\{x,y\in D:\, |x-y|>\varepsilon\}} K(x-y)f(y)g(x)dydx$$

$$= \int_D f(x)(\widetilde{T}_D g)(x)dx, \tag{7}$$

where

$$(\widetilde{T}_D g)(x) = \lim_{\varepsilon\to 0+} \int_{\{y\in D\,:\,|x-y|>\varepsilon\}} K(y-x)g(y)dy, \qquad x \in D.$$

In particular, if the kernel K is even, then

$$\int_D g(x)(T_D f)(x)dx = \int_D f(x)(T_D g)(x)dx,$$

and if it is odd, then

$$\int_D g(x)(T_D f)(x)dx = -\int_D f(x)(T_D g)(x)dx.$$

In this section we put forward an analogue of (7) for $f \in L_1(D)$.

Theorem. *Let $f \in L_1(D)$ and $g(x)$ be a bounded function on D with bounded $(\widetilde{T}_D g)(x)$ on D. Then the function $g(x)(T_D f)(x)$ is A-integrable on D and*

$$(A)\int_D g(x)(T_D f)(x)dx = \int_D f(x)(\widetilde{T}_D g)(x)dx. \tag{8}$$

In particular, if the kernel K is even, then

$$(A)\int_D g(x)(T_D f)(x)dx = \int_D f(x)(T_D g)(x)dx,$$

and if it is odd, then

$$(A)\int_D g(x)(T_D f)(x)dx = -\int_D f(x)(T_D g)(x)dx.$$

Proof. Since the A-integral satisfies the additivity property, it can be assumed that the function f is real, $f(x) \geq 0$ for any $x \in D$, and $\sup_{x\in D}\{|g(x)|, |(\widetilde{T}_D g)(x)|\} \leq 1$. For $x \notin D$ we assume that $f(x) = 0$.

Our proof will depend on a certain refinement of Besicovitch's method [8] for a direct proof of the existence of the conjugate function (this method employs only the machinery of the theory of sets of points). This method was improved by Titchmarsh [15] and Ul'yanov [16] for the study of properties of the conjugate function. It is worth noting that Besicovitch–Titchmarsh–Ul'yanov's method is applicable only to functions of one real variable (because this method relies on some facts that are valid only in the one-dimensional case). For example, it depends on the fact that any open set is a union of an at most countable number of intervals (to overcome this difficulty, we used Vitali's covering lemma). To make this method to work in the setting of functions of several variables, we have slightly improved the construction, which for simplicity of presentation is divided into three steps. We note that this method used in [6] and [7] to obtain an analogue of the Riesz equality for the Ahlfors-Beurling and Riesz transforms.

Step 1. In this part we construct and study properties of the sets G_p, L_n, L'_n, T_n and the functions $\Phi_n(x)$, $\Phi^*_n(x)$, which we shall use later.

Denote $\Phi_n(x) = f(x) - [f(x)]^n$ for $x \in D$ and $\Phi_n(x) = 0$ for $x \in R^d/D$. Then $\varepsilon_n = \int_D \Phi_n(x)dx \to 0$ as $n \to \infty$. Take $n \in N$ such that $\varepsilon_n < 1$. Let $E_n = \{x \in D : f(x) > n\}$. For any $x \in E_n$ we set

$$r_x = \sup\{r > 0 : \int_{B(x;r)} \Phi_n(y)dy = \frac{n}{2}\ \omega_d r^d\}$$

if $\{r > 0 : \int_{B(x;r)} \Phi_n(y)dy = \frac{n}{2}\ \omega_d r^d\} \neq \emptyset$, and define $r_x = 0$ otherwise, where $B(x;r)$ is an open ball with center at x and with radius r, $\omega_d = m(B(x;1)) = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ is measure of d-dimensional unit ball. Note that if $x \in E_n$ is a Lebesgue point of the function $\Phi_n(x)$, then $r_x > 0$ and, therefore, the set $E_n \backslash E'_n$ has a zero measure, where $E'_n = \{x \in E_n : r_x > 0\}$.

Consider the system of sets $\{B(x;r_x)\}_{x \in E'_n}$. It follows from the Vitali's covering lemma (see [11]) that there exists an at more countable set of points $x_k \in E'_n$, $k \in I \subset N$ such that the balls $B(x_k;r_{x_k})$, $k \in I$ are pairwise disjoint and

$$\bigcup_{x \in E'_n} B(x;r_x) \subset \bigcup_{k \in I} B(x_k;5r_{x_k}).$$

Denote

$$G_1 = B(x_1;5r_{x_1}) \backslash \bigcup_{k>1} B(x_k;r_{x_k}),$$

$$G_p = B(x_p;5r_{x_p}) \backslash \left[\bigcup_{k=1}^{p-1} G_k \bigcup \left(\bigcup_{k>p} B(x_k;r_{x_k})\right)\right], \qquad p \geq 2, \quad p \in I.$$

Then the measurable sets G_p, $p \in I$ are pairwise disjoint, and moreover,

$$B(x_p;r_{x_p}) \subset G_p \subset B(x_p;5r_{x_p}), p \in I,$$

$$E'_n \subset \bigcup_{x \in E'_n} B(x;r_x) \subset \bigcup_{p \in I} G_p = \bigcup_{p \in I} B(x_p;5r_{x_p}).$$

Denote $\Phi^*_n(x) = \frac{1}{m(G_p)} \int_{G_p} \Phi_n(y)dy$ for $x \in G_p$, $p \in I$ and $\Phi^*_n(x) = 0$ for $x \in R^d \backslash \bigcup_{p \in I} G_p$. Then for any $p \in I$ we have

$$\int_{G_p} \Phi_n(x)dx = \int_{G_p} \Phi^*_n(x)dx. \tag{9}$$

Note that, for any $x \in G_p$, $p \in I$,

$$0 \leq \Phi^*_n(x) \leq \frac{1}{m(B(x_p;r_{x_p}))} \int_{B(x_p;5r_{x_p})} \Phi_n(y)dy \leq \frac{1}{\omega_d r^d_{x_p}} \cdot \frac{n}{2}\omega_d 5^d r^d_{x_p} = \frac{5^d n}{2}.$$

Setting $L_n = \bigcup_{p\in I} G_p$, $L'_n = \bigcup_{p\in I} B(x_p; 10r_{x_p})$, we have

$$m(L_n) \le \sum_{p\in I} \omega_d 5^d r_{x_p}^d \le 5^d \cdot \frac{2}{n} \sum_{p\in I} \int_{B(x_p;r_{x_p})} \Phi_n(y)dy$$

$$\le \frac{2\cdot 5^d}{n} \int_D \Phi_n(y)dy = \frac{2\cdot 5^d \varepsilon_n}{n},$$

$$m(L'_n) \le \sum_{p\in I} \omega_d 10^d r_{x_p}^d \le \frac{2\cdot 10^d \varepsilon_n}{n}. \tag{10}$$

Let $D_n = D\backslash L'_n$. We first prove the inequality

$$\int_{D_n} |T_D(\Phi_n - \Phi_n^*)(x)|dx \le 2\cdot 10^d d^2 \omega_d \|\Omega\|_\alpha \cdot \varepsilon_n, \tag{11}$$

where

$$\|\Omega\|_\alpha = \|\Omega\|_\infty + H(\Omega;\alpha;S^{d-1}),$$

$$\|\Omega\|_\infty = \max_{x\in S^{d-1}} |\Omega(x)|,\ H(\Omega;\alpha;S^{d-1}) = \sup_{x,y\in S^{d-1}, x\ne y} \frac{|\Omega(x)-\Omega(y)|}{|x-y|^\alpha}.$$

Denote $h_n(x) = T_D(\Phi_n - \Phi_n^*)(x)$. For any $x\in D_n$ we have

$$|h_n(x)| = \left| \int_D K(x-y)[\Phi_n(y) - \Phi_n^*(y)]dy \right|$$

$$= \left| \sum_{p\in I} \int_{G_p} K(x-y)[\Phi_n(y) - \Phi_n^*(y)]dy \right|$$

$$\le \sum_{p\in I} \left| \int_{G_p} \frac{\Omega(x-y)}{|x-y|^d} \Phi_n(y)dy - \int_{G_p} \frac{\Omega(x-y)}{|x-y|^d} \Phi_n^*(y)dy \right|. \tag{12}$$

It follows from the integral mean value theorem that for any $p\in I$ there are points y_p, $y_p^* \in B(x_p; 5r_{x_p})$ such that

$$\int_{G_p} \frac{\Omega(x-y)}{|x-y|^d} \Phi_n(y)dy = \frac{\Omega(x-y_p)}{|x-y_p|^d} \int_{G_p} \Phi_n(y)dy,$$

$$\int_{G_p} \frac{\Omega(x-y)}{|x-y|^d} \Phi_n^*(y)dy = \frac{\Omega(x-y_p^*)}{|x-y_p^*|^d} \int_{G_p} \Phi_n^*(y)dy.$$

Then, using (9) and (12),

$$|h_n(x)| \le \sum_{p\in I} \left| \frac{\Omega(x-y_p)}{|x-y_p|^d} - \frac{\Omega(x-y_p^*)}{|x-y_p^*|^d} \right| \cdot \int_{G_p} \Phi_n(y)dy. \tag{13}$$

Since, for any y, $y^* \in B(x_p; 5r_{x_p})$ and $x \in D_n$,

$$\left|\frac{\Omega(x-y)}{|x-y|^d} - \frac{\Omega(x-y^*)}{|x-y^*|^d}\right|$$

$$\leq |\Omega(x-y)| \cdot \left|\frac{1}{|x-y|^d} - \frac{1}{|x-y^*|^d}\right| + \frac{1}{|x-y|^d} \cdot |\Omega(x-y) - \Omega(x-y^*)|$$

$$\leq \|\Omega\|_\infty \cdot \frac{\left||x-y^*|^d - |x-y|^d\right|}{|x-y|^d|x-y^*|^d} + \frac{1}{|x-y|^d} \cdot \left|\Omega\left(\frac{x-y}{|x-y|}\right) - \Omega\left(\frac{x-y^*}{|x-y^*|}\right)\right|$$

$$\leq \|\Omega\|_\infty \cdot |y-y^*| \sum_{k=0}^{d-1} |x-y|^{k-d}|x-y^*|^{-k-1} + \frac{H(\Omega;\alpha;S^{d-1})}{|x-y|^d} \cdot \left|\frac{x-y}{|x-y|} - \frac{x-y^*}{|x-y^*|}\right|^\alpha$$

$$\leq \frac{10d \cdot 2^{d+1} r_{x_p}}{|x-x_p|^{d+1}}\|\Omega\|_\infty + \frac{2^d 40^\alpha r_{x_p}^\alpha}{|x-x_p|^{d+\alpha}} H(\Omega;\alpha;S^{d-1}),$$

it follows from (13) that

$$|h_n(x)| \leq \sum_{p\in I}\left[\frac{10d \cdot 2^{d+1} r_{x_p}}{|x-x_p|^{d+1}}\|\Omega\|_\infty + \frac{2^d 40^\alpha r_{x_p}^\alpha}{|x-x_p|^{d+\alpha}} H(\Omega;\alpha;S^{d-1})\right] \cdot \int_{G_p} \Phi_n(y)dy$$

$$\leq \sum_{p\in I}\left[\frac{10d \cdot 2^{d+1} r_{x_p}}{|x-x_p|^{d+1}}\|\Omega\|_\infty + \frac{2^d 40^\alpha r_{x_p}^\alpha}{|x-x_p|^{d+\alpha}} H(\Omega;\alpha;S^{d-1})\right] \cdot \left(\frac{n}{2}\omega_d 5^d r_{x_p}^d\right)$$

$$= \sum_{p\in I}\left[\frac{10^{d+1} nd\omega_d r_{x_p}^{d+1}}{|x-x_p|^{d+1}}\|\Omega\|_\infty + \frac{5 \cdot 10^{d-1} 40^\alpha n\omega_d r_{x_p}^{d+\alpha}}{|x-x_p|^{d+\alpha}} H(\Omega;\alpha;S^{d-1})\right]$$

Therefore,

$$\int_{D_n} |h_n(x)|dx \leq 10^{d+1} nd\omega_d \|\Omega\|_\infty \sum_{p\in I} r_{x_p}^{d+1} \int_{D_n} \frac{dx}{|x-x_p|^{d+1}}$$

$$+5 \cdot 10^{d-1} 40^\alpha n\omega_d H(\Omega;\alpha;S^{d-1}) \sum_{p\in I} r_{x_p}^{d+\alpha} \int_{D_n} \frac{dx}{|x-x_p|^{d+\alpha}}$$

$$\leq 10^{d+1} nd\omega_d \|\Omega\|_\infty \sum_{p\in I} r_{x_p}^{d+1} \int_{\{x:|x-x_p|\geq 10r_{x_p}\}} \frac{dx}{|x-x_p|^{d+1}}$$

$$+5 \cdot 10^{d-1} 40^\alpha n\omega_d H(\Omega;\alpha;S^{d-1}) \sum_{p\in I} r_{x_p}^{d+\alpha} \int_{\{x:|x-x_p|\geq 10r_{x_p}\}} \frac{dx}{|x-x_p|^{d+\alpha}}$$

$$= 5 \cdot 10^{d-1} nd\omega_d^2 \left[2d\|\Omega\|_\infty + 4^\alpha H(\Omega;\alpha;S^{d-1})\right] \cdot \sum_{p\in I} r_{x_p}^d$$

$$\leq 2 \cdot 10^d d^2 \omega_d \|\Omega\|_\alpha \cdot \varepsilon_n$$

proving inequality (11).

We represent the function $f(x)$ in the form

$$f(x) = [f(x)]^n + \Phi_n^*(x) + [\Phi_n - \Phi_n^*](x). \tag{14}$$

Step 2. In this part we prove the equality

$$\lim_{n\to\infty} \int_{D_n} g(x)(T_D f)(x)dx = \int_D f(x)(\widetilde{T}_D g)(x)dx. \tag{15}$$

Consider the integral

$$\int_{D_n} g(x)(T_D f)(x)dx$$

$$= \int_{D_n} g(x)\{(T_D[f]^n)(x) + (T_D\Phi_n^*)(x) + T_D(\Phi_n - \Phi_n^*)(x)\}dx$$

$$= \int_{D_n} g(x)(T_D[f]^n)(x)dx + \int_{D_n} g(x)(T_D\Phi_n^*)(x)dx$$

$$+ \int_{D_n} g(x)T_D(\Phi_n - \Phi_n^*)(x)dx = S_1 + S_2 + S_3. \tag{16}$$

By (7), we have

$$S_1 = \int_{D_n} g(x)(T_D[f]^n)(x)dx$$

$$= \int_D g(x)(T_D[f]^n)(x)dx - \int_{L_n'} g(x)(T_D[f]^n)(x)dx$$

$$= \int_D [f(x)]^n(\widetilde{T}_D g)(x)dx - \int_{L_n'} g(x)(T_D[f]^n)(x)dx = S_1^{(1)} + S_1^{(2)}.$$

Since

$$|S_1^{(2)}| = \left|\int_{L_n'} g(x)(T_D[f]^n)(x)dx\right| \le \int_{L_n'} |(T_D[f]^n)(x)|dx$$

$$\le \left[m(L_n')\cdot\int_D |(T_D[f]^n)(x)|^2 dx\right]^{1/2}$$

$$\le C_2\left[m(L_n')\cdot\int_D ([f(x)]^n)^2 dx\right]^{1/2}$$

$$\le C_2\left[n\cdot m(L_n')\cdot\int_D f(x)dx\right]^{1/2},$$

it follows from (10) that

$$\lim_{n\to\infty} S_1 = \lim_{n\to\infty}\int_D [f(x)]^n(\widetilde{T}_D g)(x)dx = \int_D f(x)(\widetilde{T}_D g)(x)dx. \tag{17}$$

For the integral S_2 we also have

$$S_2 = \int_{D_n} g(x)(T_D\Phi_n^*)(x)dx$$
$$= \int_D g(x)(T_D\Phi_n^*)(x)dx - \int_{L'_n} g(x)(T_D\Phi_n^*)(x)dx$$
$$= \int_D \Phi_n^*(x)(\widetilde{T}_D g)(x)dx - \int_{L'_n} g(x)(T_D\Phi_n^*)(x)dx = S_2^{(1)} + S_2^{(2)}.$$

The following estimates are valid:

$$|S_2^{(1)}| = \left|\int_D \Phi_n^*(x)(\widetilde{T}_D g)(x)dx\right| \le \int_D |\Phi_n^*(x)(\widetilde{T}_D g)(x)|dx$$
$$\le \int_D \Phi_n^*(x)dx = \int_D \Phi_n(x)dx = \varepsilon_n,$$
$$|S_2^{(2)}| = \left|\int_{L'_n} g(x)(T_D\Phi_n^*)(x)dx\right| \le \int_{L'_n} |(T_D\Phi_n^*)(x)|dx$$
$$\le \left[m(L'_n)\cdot\int_D |(T_D\Phi_n^*)(x)|^2dx\right]^{1/2} \le C_2\left[m(L'_n)\cdot\int_D (\Phi_n^*(x))^2dx\right]^{1/2}$$
$$\le C_2\left[\frac{5^d}{2}n\cdot m(L'_n)\cdot\int_D \Phi_n^*(x)dx\right]^{1/2} = C_2\left[\frac{5^d}{2}n\cdot m(L'_n)\cdot\varepsilon_n\right]^{1/2}.$$

Then it follows from (10) that

$$\lim_{n\to\infty} S_2 = 0. \tag{18}$$

To estimate the integral S_3, we apply inequality (11). We have

$$|S_3| = \left|\int_{D_n} g(x)T_D(\Phi_n - \Phi_n^*)(x)dx\right| \le \int_{D_n} |g(x)T_D(\Phi_n - \Phi_n^*)(x)|dx$$
$$\le \int_{D_n} |T_D(\Phi_n - \Phi_n^*)(x)|dx \le 2\cdot 10^d d^2\omega_d\|\Omega\|_\alpha\cdot\varepsilon_n.$$

This implies the equality

$$\lim_{n\to\infty} S_3 = 0. \tag{19}$$

Now (15) follows from equalities (16), (17), (18) and (19).

Step 3. In this part we prove the equality

$$(A)\int_D g(x)(T_Df)(x)dx = \lim_{n\to\infty}\int_{D_n} g(x)(T_Df)(x)dx. \tag{20}$$

Consider the difference of integrals

$$\int_{D_n} g(x)(T_Df)(x)dx - \int_D [g(x)(T_Df)(x)]^n dx$$

$$= -\int_{L'_n} [g(x)(T_D f)(x)]^n dx$$

$$+ \int_{D_n} \{g(x)(T_D f)(x) - [g(x)(T_D f)(x)]^n\} \, dx = S^{(1)} + S^{(2)}. \tag{21}$$

From the inequality $|S^{(1)}| \le n \cdot m(L'_n)$ it follows that

$$\lim_{n\to\infty} S^{(1)} = 0. \tag{22}$$

Denote $\sigma_n = \{x \in D : |g(x)(T_D f)(x)| > n\}$.

Since $m\{x \in D : |(T_D f)(x)| > n\} = o(1/n)$, $n \to \infty$, we have $m(\sigma_n) = o(1/n)$, $n \to \infty$. Using (11) and (14), we obtain

$$\left|S^{(2)}\right| \le \int_{D_n \bigcap \sigma_n} |g(x)(T_D f)(x)| dx \le \int_{D_n \bigcap \sigma_n} |(T_D f)(x)| dx$$

$$\le \int_{\sigma_n} |(T_D [f]^n)(x)| dx + \int_{\sigma_n} |(T_D \Phi_n^*)(x)| dx + \int_{D_n} |T_D(\Phi_n - \Phi_n^*)(x)| dx$$

$$\le \left[m(\sigma_n) \cdot \int_D |(T_D [f]^n)(x)|^2 dx \right]^{1/2}$$

$$+ \left[m(\sigma_n) \cdot \int_D |(T_D \Phi_n^*)(x)|^2 dx \right]^{1/2} + 2 \cdot 10^d d^2 \omega_d \|\Omega\|_\alpha \cdot \varepsilon_n$$

$$\le C_2 \left[m(\sigma_n) \cdot \int_D ([f(x)]^n)^2 dx \right]^{1/2}$$

$$+ C_2 \left[m(\sigma_n) \cdot \int_D (\Phi_n^*(x))^2 dx \right]^{1/2} + 2 \cdot 10^d d^2 \omega_d \|\Omega\|_\alpha \cdot \varepsilon_n$$

$$\le C_2 \left[n \cdot m(\sigma_n) \cdot \int_D f(x) dx \right]^{1/2}$$

$$+ C_2 \left[\frac{5^d}{2} n \cdot m(\sigma_n) \cdot \int_D \Phi_n^*(x) dx \right]^{1/2} + 2 \cdot 10^d d^2 \omega_d \|\Omega\|_\alpha \cdot \varepsilon_n.$$

It follows that

$$\lim_{n\to\infty} S^{(2)} = 0. \tag{23}$$

Now equality (20) follows from equalities (21), (22) and (23).

From the equalities (15) and (20) we obtain (8). ◀

Corollary. *Let Ω be an even, homogeneous function of degree 0, continuously differentiable on $R^d \setminus 0$ and with zero integral on the unit sphere. If $f \in L_1(D)$ and the boundary of the domain D is a Lyapunov surface, then the function $(T_D f)(x)$ is A-integrable on D.*

Proof. Indeed, if the boundary of D is a Lyapunov surface, then taking $g(z) \equiv 1$, we see that the function $(T_D g)(x)$ is also bounded, and it follows from Theorem that the function $(T_D f)(x)$ is A-integrable on D and

$$(A)\int_D (T_D f)(x)dx = \int_D f(x)(T_D 1)(x)dx.$$

◄

References

1. Aleksandrov A.B. A-integrability of the boundary values of harmonic functions. *Math. Notes*, 1981, **30** (1), pp. 515-523.
2. Aliev R.A. $N^{\pm}$-integrals and boundary values of Cauchy-type integrals of finite measures. *Sb. Math.*, 2014, **205** (7), pp. 913-935.
3. Aliev R.A. On properties of Hilbert transform of finite complex measures. *Complex Anal. Oper. Theory*, 2016, **10** (1), pp. 171-185.
4. Aliev R.A. Riesz's equality for the Hilbert transform of the finite complex measures. *Azerb. J. Math.*, 2016, **6** (1), pp. 126-135.
5. Aliev R.A. Representability of Cauchy-type integrals of finite complex measures on the real axis in terms of their boundary values. *Complex Var. Elliptic Equ.*, 2017, **62** (4), pp. 536-553.
6. Aliev R.A., Nebiyeva Kh.I. The A-integral and restricted Ahlfors-Beurling transform. *Integral Transforms Spec. Funct.*, 2018, **29** (10), pp. 820-830.
7. Aliev R.A., Nebiyeva Kh.I. The A-integral and restricted Riesz transform. *Constr. Math. Anal.*, 2020, **3** (3), pp. 104-112.
8. Besicovitch A.S. On a general metric property of summable functions. *J. London Math. Soc.*, 1926, **1** (2), pp. 120-128.
9. Efimova M.P. On the properties of the Q-integral. *Math. Notes*, 2011, **90** (3-4), pp. 322-332.
10. Efimova M.P. The sufficient condition for integrability of a generalized Q-integral and points of integrability. *Moscow Univ. Math. Bull.*, 2015, **70** (4), pp. 181-184.
11. Evans L.C., Gariepy R.F. *Measure Theory and Fine Properties of Functions.* CRC Press, Boca Raton, 1992.
12. Salimov T.S. The A-integral and boundary values of analytic functions. *Math. USSR-Sb.*, 1989, **64** (1), pp. 23-39.
13. Skvortsov V.A. A-integrable martingale sequences and Walsh series. *Izv. Math.*, 2001, **65** (3), pp. 607-615.
14. Stein E.M. *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, 1970.
15. Titchmarsh E.C. On conjugate functions. *Proc. London Math. Soc.*, 1928, **29** (1), pp. 49-80.
16. Ul'yanov P.L. The A-integral and conjugate functions. *Moskov. Gos. Univ. Uč. Zap. Mat.*, 1956, **181**, pp. 139-157 (in Russian).
17. Ul'yanov P.L. Integrals of Cauchy type. Twelve Papers on Approximations and Integrals. *Amer. Math. Soc., Trans.*, 1965, **2** (44), pp. 129-150.