Baku Mathematical Journal
202X, Vol. XX, No X, PP. XX-XX
https://doi.org/10.32010/j.bmj.202X.XX

REPRESENTATION OF SOLUTIONS OF $\mathcal{M}$-SPECTRAL PROBLEM FOR DIFFUSION OPERATOR

E. BAS*, **M. KARAOGLAN**, **E. PANAKHOV**

Received: date / Revised: date / Accepted: date

In memory of M. G. Gasymov on his 85th birthday

Abstract. *In this article, we offer the $\mathcal{M}$-Sturm Liouville problem for the diffusion operator depending on initial condition. We obtain the representation of the solution of the $\mathcal{M}$-Sturm Liouville problem for diffusion operator through the $\mathcal{M}$-Laplace transform. The purpose of this article is to advantage demonstrate a more generalized version of the representation of the solution of the Sturm Liouville problem for diffusion operator in classical analysis.*

Keywords: $\mathcal{M}$-derivative, diffusion operator, $\mathcal{M}$-Laplace transform

Mathematics Subject Classification (2020): 34A08, 34L05

1. Motivation

Fractional calculus is any complex or real order theory of integrals and derivatives that combines and also generalizes the concepts of integer order derivatives and -n-fold integrals [9]. The opinion of this field first emerged in the 17th century in a letter from Leibniz to L' Hospital [9]. Fractional calculus has been the focus of study for many mathematicians [8]. Fractional diffusion equations have been examined in several different physical situations [2]. These equations are broadly applicable because there are numerous scenarios in which they prove appropriate [2]. Tuan has been demonstrated that only a

* Corresponding author.

Erdal Bas
University of Firat, Elazig, Türkiye
E-mail: erdalmat@yahoo.com

Merve Karaoglan
University of Firat, Elazig, Türkiye
E-mail: karaoglanmerve24@gmail.com

Etibar Panakhov
Institute of Applied Mathematics at Baku State University, Baku, Azerbaijan
E-mail: epenahov@hotmail.com

finite number of calculations are required in the limit to uniquely obtain the diffusion coefficient of the one-dimensional fractional diffusion equation, given appropriate initial distributions [11]. In physics, the problem of expressing the interactions among colliding particles is of immense interest [14]. Jaulent and Jean have been discovered that two particles that do not rotate around their axis with the collision [6]. They have been accepted that the scattering matrix s, indicated by $S(E)$ and the bound energy, denoted by E_n, defined for all energies $E > 0$, were known exactly from collision tests [6]. Jaulent and Jean first have been tried to find a radial stationary potential function $V(x)$ for $x \geq 0$ [6]. This potential function will give E_n and $S(E)$ $E > 0$ given by the radial s-wave Schrödinger equation [6]. This equation is expressed in the center of mass of the two particles and in the usual reduced variables as follows [6]:

$$y'' + [E - V(X)]\, y = 0, \quad x \geq 0.$$

The above equation reduces to the Klein Gordon s-wave equation [10]. The Klein-Gordon equation is recognized as one of the most important in quantum field theory [13]. The equation is commonly used to describe dispersive wave phenomena [13]. Consider the following form of the Klein-Gordon equation [10]:

$$\left[\left(i\frac{\partial}{\partial t} - e\phi\right)^2 - \left(\frac{1}{i}\nabla - e\overrightarrow{A}\right)^2\right]\Psi = m^2\Psi,$$

$$\psi''(r) + \left[V^2 - 2EV\right]\psi(r) = -K^2\psi.$$

The Klein-Gordon equation reduces to the diffusion equation above [14]. Bas, has been studied inverse nodal problem for the fractional diffusion equation [2]. Nabiev and Guseinov have been studied inverse problems for the diffusion operator on a finite interval [4]. Koyunbakan and Panakhov, have been proved that while q potential function for the diffusion operator in a finite range is determined in the range $\left[\frac{\pi}{2}, \pi\right]$ a single spectrum is sufficient to determine the potential function in the rest of the range [7]. In 2017, Sousa and Oliveira found a parameter that satisfies the properties of the integer order calculus and an $\mathcal{M}$-derivative containing the Mittag-Leffler function [12]. In later years, has attracted the interest of many scientists, it has been the subject of various publications. A unique method that facilitates the solutions of differential equations by transforming them into algebraic equations is the Laplace transform method [9]. Jarad and Abdeljawad in have been submitted the generalized Laplace transform for the generalized fractional integrals and derivatives [5]. In this paper, we deal with the $\mathcal{M}$-Sturm Liouville problem for diffusion operator. Section 2 is the structure stone of $\mathcal{M}$-derivative. Also this section, we exhibit somewhat tools obligatory for our study, such as the $\mathcal{M}$-derivative of specific functions, the $\mathcal{M}$-Laplace transform. This method is used as an alternative method for solving differential equations. The $\mathcal{M}$-Laplace transform, is a powerful guiding tool that facilitates the solution of problems in many field and provides practical information. Section 3 is the main backbone of our study. We give the representation of the solution of the $\mathcal{M}$-Sturm Liouville problem for diffusion operator by the $\mathcal{M}$-Laplace transform. Moreover, the inimitableness of the solution has been proven here. Section 4 presents a exhaustive discussion supported by graphs for diverse values of α, γ, h and λ. The recent section includes considerable explanations for our main results.

2. General Properties of Method

Before coming to the main results, we offer some important definitions, theorems and properties about the $\mathcal{M}$-derivative.

Definition 1. *[3]. Let $f:[0,\infty]\to\mathbb{R}$ function be defined. The $\mathcal{M}$-derivative for $0<\beta\le 1$ is defined as follows*

$$D_{\mathcal{M}}^{\beta,\gamma}f(t)=\lim_{\varepsilon\to 0}\frac{f\left(tE_{\gamma}\left(\varepsilon t^{-\beta}\right)\right)-f(t)}{\varepsilon},$$

where $E_{\gamma}\left(\varepsilon t^{-\beta}\right)$ is the Mittag-Leffler function [13].

Theorem 1. *[3] If $0<\alpha\le 1$, $\gamma>0$, $a,b\in\mathbb{R}$ and f,g are α-differentiable functions at a $t>0$ point, then*

1. $D_{\mathcal{M}}^{\alpha,\gamma}(c)=0$, *$c$ is a constant*;
2. $D_{\mathcal{M}}^{\alpha,\gamma}(f.g)(t)=f(t)\,D_{\mathcal{M}}^{\alpha,\gamma}g(t)+g(t)\,D_{\mathcal{M}}^{\alpha,\gamma}f(t)$;
3. $D_{\mathcal{M}}^{\alpha,\gamma}\left(\frac{f}{g}\right)(t)=\frac{g(t)D_{\mathcal{M}}^{\alpha,\gamma}f(t)-f(t)D_{\mathcal{M}}^{\alpha,\gamma}g(t)}{[g(t)]^2}$;
4. $D_{\mathcal{M}}^{\alpha,\gamma}(fog)(t)=f'(g(t))\,D_{\mathcal{M}}^{\alpha,\gamma}g(t)$.

We present now a few definitions and theorems required for our main result.

Theorem 2. *[3] $\mathcal{M}$-derivative of some important functions are as follows:*

i. $D_{\mathcal{M}}^{\alpha,\gamma}(\sin(ct)\,)=\frac{ct^{1-\alpha}\cos(ct)}{\Gamma(\gamma+1)}$, $\quad c\in\mathbb{R}$;

ii. $D_{\mathcal{M}}^{\alpha,\gamma}(\cos(ct)\,)=-\frac{ct^{1-\alpha}\sin(ct)}{\Gamma(\gamma+1)}$, $\quad c\in\mathbb{R}$;

iii. $D_{\mathcal{M}}^{\alpha,\gamma}\left(e^{bt}\right)=\frac{bt^{1-\alpha}e^{bt}}{\Gamma(\gamma+1)}$, $\quad b\in\mathbb{R}$;

iv. $D_{\mathcal{M}}^{\alpha,\gamma}\left(t^{k}\right)=\frac{kt^{k-\alpha}}{\Gamma(\gamma+1)}$, $\quad k\in\mathbb{R}$.

Also, note that the following functions in terms of $\mathcal{M}$-derivative [2]:

- $D_{\mathcal{M}}^{\alpha,\gamma}\left(e^{\frac{\Gamma(\gamma+1)t^{\alpha}}{\alpha}}\right)=\left(e^{\frac{\Gamma(\gamma+1)t^{\alpha}}{\alpha}}\right)$;
- $D_{\mathcal{M}}^{\alpha,\gamma}\left(\sin\frac{\Gamma(\gamma+1)t^{\alpha}}{\alpha}\right)=\left(\cos\frac{\Gamma(\gamma+1)t^{\alpha}}{\alpha}\right)$;
- $D_{\mathcal{M}}^{\alpha,\gamma}\left(\cos\frac{\Gamma(\gamma+1)t^{\alpha}}{\alpha}\right)=-\left(\sin\frac{\Gamma(\gamma+1)t^{\alpha}}{\alpha}\right)$.

Definition 2. *[3] n-order linear non-homogeneous differential equation in terms of the $\mathcal{M}$-derivative is given as*

$$a_0D_{\mathcal{M}}^{n\alpha,\gamma}y+a_1D_{\mathcal{M}}^{(n-1)\alpha,\gamma}y+\cdots+a_{n-1}D_{\mathcal{M}}^{\alpha,\gamma}y+a_ny=x(t),$$

where $0<\alpha<1$, $a_0\neq 0$, $D_{\mathcal{M}}^{n\alpha,\gamma}y=D_{\mathcal{M}}^{\alpha,\gamma}\,D_{\mathcal{M}}^{\alpha,\gamma}\ldots\,D_{\mathcal{M}}^{\alpha,\gamma}$ and a_0, $a_1,\ldots,a_n$ are constants or variables.

If the function y is n-times differentiable, there are n-independent solutions y_1, $y_2,\ldots,y_n$ for the homogeneous differential equation

$$a_0D_{\mathcal{M}}^{n\alpha,\gamma}y+a_1D_{\mathcal{M}}^{(n-1)\alpha,\gamma}y+\cdots+a_{n-1}D_{\mathcal{M}}^{\alpha,\gamma}y+a_ny=0.$$

Now, we introduce the $\mathcal{M}$-Laplace transform in order to solve some kind of differential equations with arbitrary order. We representation the $\mathcal{M}$-Laplace transform of some important functions.

Definition 3. *[1] Let $f:[a,\infty)\to\mathbb{R}$, $a\in\mathbb{R}$, $\gamma>0$ and $0<\rho\leq 1$, then the $\mathcal{M}$-Laplace transform is defined as below*

$$\mathcal{L}^{a}_{\rho,\gamma}\left\{f(t)\right\}(s)=F^{a}_{\rho,\gamma}(s)=\Gamma(\gamma+1)\int_{a}^{\infty}e^{-s\frac{\Gamma(\gamma+1)(t-a)^{\rho}}{\rho}}f(t)\,(t-a)^{\rho-1}dt, \qquad (1)$$

where $(t-a)^{\rho-1}dt=d_{\rho}t$.

We define the $\mathcal{M}$-Laplace transform of certain functions [1]:

- ► $\mathcal{L}_{\rho,\gamma}\left\{t^{k}\right\}(s)=\frac{\Gamma\left(1+\frac{k}{\rho}\right)\left(\frac{\rho}{\Gamma(\gamma+1)}\right)^{k/\rho}}{s^{\frac{k+\rho}{\rho}}}\ s>0, \qquad k\in\mathbb{R};$
- ► $\mathcal{L}_{\rho,\gamma}\left\{\sin\left(b\Gamma(\gamma+1)\frac{t^{\rho}}{\rho}\right)\right\}(s)=\frac{b}{b^{2}+s^{2}}, \qquad b\in\mathbb{R};$
- ► $\mathcal{L}_{\rho,\gamma}\left\{\cos\left(b\Gamma(\gamma+1)\frac{t^{\rho}}{\rho}\right)\right\}(s)=\frac{s}{b^{2}+s^{2}}, \qquad b\in\mathbb{R};$
- ► $\mathcal{L}_{\rho,\gamma}\left\{\Gamma(\gamma+1)\frac{t^{\rho}}{\rho}e^{c\Gamma(\gamma+1)\frac{t^{\rho}}{\rho}}\right\}(s)=\frac{1}{(s-c)^{2}}, \qquad c\in\mathbb{R}.$

Theorem 3. *[1] Let f and g are continuous and real valued functions. Also, both have Laplace transform such that $\mathcal{L}^{\;a}_{\rho,\gamma}\left\{f(t)\right\}(s)=F^{a}_{\rho,\gamma}(s)$, $\mathcal{L}^{a}_{\rho,\gamma}\left\{g(t)\right\}(s)=G^{a}_{\rho,\gamma}(s)$ and $F^{a}_{\rho,\gamma}(s)=G^{a}_{\rho,\gamma}(s)$. Then $f(t)=g(t)$.*

Theorem 4. *[1] Assume that $f:[a,\infty)\to\mathbb{R}$, $a\in\mathbb{R}$, $0<\rho\leq 1$ and for $t\geq t_0$ there exist the constants M, b, t_0 such that $|f(t)|\leq Me^{b\Gamma(\gamma+1)\frac{(t-a)^{\rho}}{\rho}}$. Then*

$$\mathcal{L}^{a}_{\rho,\gamma}\left\{D^{\rho,\gamma}_{\mathcal{M}}f(t)\right\}=s\mathcal{L}^{a}_{\rho,\gamma}\left\{f(t)\right\}-f(a)$$

and in a general form we write

$$\mathcal{L}^{a}_{\rho,\gamma}\left\{D^{(n)\rho,\gamma}_{\mathcal{M}}f(t)\right\}=s^{n}\mathcal{L}^{a}_{\rho,\gamma}\left\{f(t)\right\}-s^{n-1}f(a)-s^{n-2}D^{\rho,\gamma}_{\mathcal{M}}f(a)$$
$$-\cdots-sD^{(n-2)\rho,\gamma}_{\mathcal{M}}f(a)-D^{(n-1)\rho,\gamma}_{\mathcal{M}}f(a).$$

Definition 4. *[1] Let $f(t)$ and $g(t)$ are piecewise continuous functions and have exponential order, then the convolution of f and g in frame of the $\mathcal{M}$-derivative is defined by*

$$(f*g)(t)=\Gamma(\gamma+1)\int_{a}^{t}f(\tau)\,g(a+((t-a)^{\rho}-(\tau-a)^{\rho})^{1/\rho}\,(\tau-a)^{\rho-1}d\tau. \qquad (2)$$

Theorem 5. *[1] Let $f,g:[a,\infty)\to\mathbb{R}$, $a\in\mathbb{R}$, $0<\rho\leq 1$, $\gamma>0$, $s>0$ and there exist $f(t)$ and $g(t)$ such that $F^{a}_{\rho,\gamma}(s)=\mathcal{L}^{\;a}_{\rho,\gamma}\left\{f(t)\right\}$ and $G^{a}_{\rho,\gamma}(s)=\mathcal{L}^{a}_{\rho,\gamma}\left\{g(t)\right\}$, then we have the following relation*

$$\mathcal{L}^{a}_{\rho,\gamma}\left\{f*g\right\}(t)=F^{a}_{\rho,\gamma}(s)\;G^{a}_{\rho,\gamma}(s),$$

where $(f*g)$ is the convolution of f and g.

3. Main Results

In the current part, our aim is to show that the $\mathcal{M}$-Sturm Liouville problem for diffusion operator more general version of the Sturm Liouville problem for diffusion operator in classical analysis.

In this section, we consider the $\mathcal{M}$-Sturm Liouville problem for diffusion operator

$$-D_{\mathcal{M}}^{\alpha,\gamma} D_{\mathcal{M}}^{\alpha,\gamma} y(x) + [q(x) + 2\lambda p(x)] y(x) = \lambda^2 y(x),$$

where the function $q(x)$ real and continuous $q(x)$ and $p(x)$ also $0 < \alpha \leq 1$. Considering the basic opinion above, we offer the following significant theorem:

Theorem 6. *We consider the representation of the solution $\mathcal{M}$- Sturm Liouville problem for diffusion operator*

$$-D_{\mathcal{M}}^{\alpha,\gamma} D_{\mathcal{M}}^{\alpha,\gamma} y(x) + [q(x) + 2\lambda p(x)] y(x) = \lambda^2 y(x), \tag{3}$$

where the initial conditions are $y(0,\lambda) = 1$, $D_{\mathcal{M}}^{\alpha,\gamma} y(0,\lambda) = -h$ and $h = -\cot\alpha$,

$$y(x,\lambda) = \cos\left(\lambda \Gamma(\gamma+1) \frac{x^\alpha}{\alpha}\right) - \frac{h}{\lambda} \sin\left(\lambda \Gamma(\gamma+1) \frac{x^\alpha}{\alpha}\right) +$$

$$+\frac{\Gamma(\gamma+1)}{\lambda} \int_0^x \sin\left(\lambda \Gamma(\gamma+1)\left(\frac{x^\alpha}{\alpha} - \frac{\tau^\alpha}{\alpha}\right)\right) [q(\tau) + 2\lambda p(\tau)] y(\tau,\lambda)\, d_\alpha \tau.$$

Proof. Let is get the representation of solution the $\mathcal{M}$- Sturm Liouville problem for diffusion operator by using the $\mathcal{M}$-Laplace transform (3). If we apply the $\mathcal{M}$-Laplace transform to both sides of problem (3),

$$-\mathcal{L}^{\alpha,\gamma}\left\{D_{\mathcal{M}}^{\alpha,\gamma} D_{\mathcal{M}}^{\alpha,\gamma} y(x)\right\} + \mathcal{L}^{\alpha,\gamma}\left\{[q(x) + 2\lambda p(x)] y(x)\right\} = \mathcal{L}^{\alpha,\gamma}\left\{\lambda^2 y(x)\right\} \tag{4}$$

it becomes. By performing the necessary operations

$$-\mathcal{L}^{\alpha,\gamma}\left\{D_{\mathcal{M}}^{2\alpha,\gamma} y(x)\right\} = -\left[s^2 Y_{\alpha,\gamma}(s) - s y(0) - D_{\mathcal{M}}^{\alpha,\gamma} y(0,\lambda)\right]. \tag{5}$$

If we write the initial conditions $y(0,\lambda) = 1$, $D_{\mathcal{M}}^{\alpha,\gamma} y(0,\lambda) = -h$ and $h = -\cot\alpha$ in equation (5),

$$-\mathcal{L}^{\alpha,\gamma}\left\{D_{\mathcal{M}}^{2\alpha,\gamma} y(x)\right\} = -s^2 Y_{\alpha,\gamma}(s) + s - h \tag{6}$$

is obtained. Using the formula (1), we write it as

$$\mathcal{L}^{\alpha,\gamma}\left\{[q(x) + 2\lambda p(x)] y(x)\right\} = \Gamma(\gamma+1) \int_0^\infty e^{-s\frac{\Gamma(\gamma+1)x^\alpha}{\alpha}} [q(x) + 2\lambda p(x)] y(x)\, d_\alpha x. \tag{7}$$

It is written

$$\mathcal{L}^{\alpha,\gamma}\left\{\lambda^2 y(x)\right\} = \lambda^2 Y_{\alpha,\gamma}(s) \tag{8}$$

in format. Substitute equations (6), (7), (8) in equation (4) to get

$$\left[-s^2 Y_{\alpha,\gamma}(s) + s - h\right] + \Gamma(\gamma+1) \int_0^\infty e^{-s\frac{\Gamma(\gamma+1)x^\alpha}{\alpha}} [q(x) + 2\lambda p(x)] y(x)\, d_\alpha x = \lambda^2 Y_{\alpha,\gamma}(s),$$

$$Y_{\alpha,\gamma}(s)\left[s^2+\lambda^2\right]=s-h+\Gamma(\gamma+1)\int_0^\infty e^{-s\frac{\Gamma(\gamma+1)x^\alpha}{\alpha}}\left[q(x)+2\lambda p(x)\right]y(x)\,d_\alpha x. \quad (9)$$

Dividing both sides of (9) by $\left[s^2+\lambda^2\right]$, we get

$$Y_{\alpha,\gamma}(s)=\frac{s}{\lambda^2+s^2}-\frac{h}{\lambda^2+s^2}+\frac{\Gamma(\gamma+1)}{\lambda^2+s^2}\int_0^\infty e^{-s\frac{\Gamma(\gamma+1)x^\alpha}{\alpha}}\left[q(x)+2\lambda p(x)\right]y(x)\,d_\alpha x. \quad (10)$$

Applying the inverse $\mathcal{M}$-Laplace transform to both sides of (10) we get

$$\mathcal{L}^{-1}_{\alpha,\gamma}\left\{Y_{\alpha,\gamma}(s)\right\}=\mathcal{L}^{-1}_{\alpha,\gamma}\left\{\frac{s}{\lambda^2+s^2}\right\}-\mathcal{L}^{-1}_{\alpha,\gamma}\left\{\frac{h}{\lambda^2+s^2}\right\}+$$

$$+\mathcal{L}^{-1}_{\alpha,\gamma}\left\{\frac{\Gamma(\gamma+1)}{\lambda^2+s^2}\int_0^\infty e^{-s\frac{\Gamma(\gamma+1)x^\alpha}{\alpha}}\left[q(x)+2\lambda p(x)\right]y(x)\,d_\alpha x\right\}. \quad (11)$$

Using the formula of $\mathcal{L}^{-1}_{\alpha,\gamma}\left\{\frac{s}{b^2+s^2}\right\}=\cos\left(b\Gamma(\gamma+1)\frac{x^\alpha}{\alpha}\right)$, we get

$$\mathcal{L}^{-1}_{\alpha,\gamma}\left\{\frac{s}{\lambda^2+s^2}\right\}=\cos\left(\lambda\Gamma(\gamma+1)\frac{x^\alpha}{\alpha}\right). \quad (12)$$

Using the formula of $\mathcal{L}^{-1}_{\alpha,\gamma}\left\{\frac{b}{b^2+s^2}\right\}=sin\left(b\Gamma(\gamma+1)\frac{x^\alpha}{\alpha}\right)$, we obtain

$$\mathcal{L}^{-1}_{\alpha,\gamma}\left\{-\frac{h}{\lambda^2+s^2}\right\}=-\frac{h}{\lambda}sin\left(\lambda\Gamma(\gamma+1)\frac{x^\alpha}{\alpha}\right). \quad (13)$$

The $\mathcal{L}^{-1}_{\alpha,\gamma}\left\{\frac{\Gamma(\gamma+1)}{\lambda^2+s^2}\int_0^\infty e^{-s\frac{\Gamma(\gamma+1)x^\alpha}{\alpha}}\left[q(x)+2\lambda p(x)\right]y(x)\,d_\alpha x\right\}$ expression can be written in

$$\frac{\Gamma(\gamma+1)}{\lambda}sin\left(\lambda\Gamma(\gamma+1)\frac{x^\alpha}{\alpha}\right)*\left[q(x)+2\lambda p(x)\right]y(x,\ \lambda)=$$

$$=\frac{\Gamma(\gamma+1)}{\lambda}\int_0^x sin\left(\lambda(\Gamma(\gamma+1))\left(\frac{x^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right)\left[q(\tau)+2\lambda p(\tau)\right]\,y(\tau,\lambda)d_\alpha\tau \quad (14)$$

by using the convolution property in (2). If we substitute the expressions (12), (13), (14) in equation (11), we get the representation of the solution in form

$$y(x,\lambda)=\cos\left(\lambda\Gamma(\gamma+1)\frac{x^\alpha}{\alpha}\right)-\frac{h}{\lambda}sin\left(\lambda\Gamma(\gamma+1)\frac{x^\alpha}{\alpha}\right)$$

$$+\frac{\Gamma(\gamma+1)}{\lambda}\int_0^x sin\left(\lambda(\Gamma(\gamma+1))\left(\frac{x^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right)\left[q(\tau)+2\lambda p(\tau)\right]\,y(\tau,\lambda)d_\alpha\tau. \quad (15)$$

◄

4. Visual Results and Discussions

In this part, we deal with diverse data of the $\mathcal{M}$-Sturm Liouville problem for diffusion operator. In the light of the observations made, we point out that it is the general version of the classical Sturm Liouville problem for diffusion operator. The graphs below clearly show us the behavioral representation of the solution is obtained for this purpose. We ensure details characterization supported by graphs for different values of α, γ, h and λ.

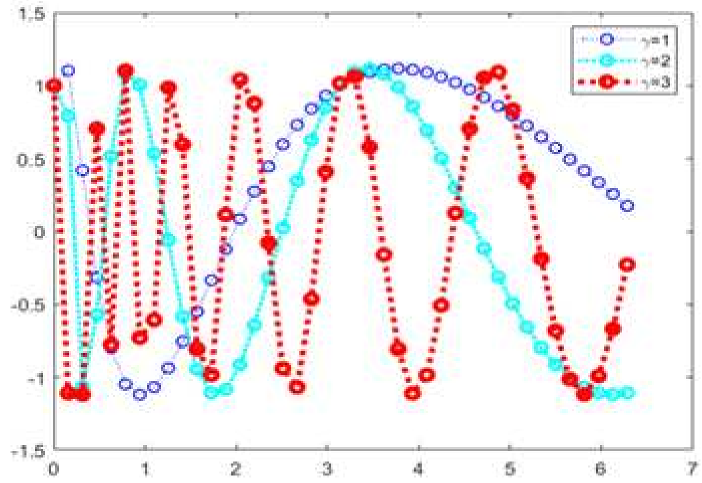

Fig. 1. Representation of solution for various γ values.

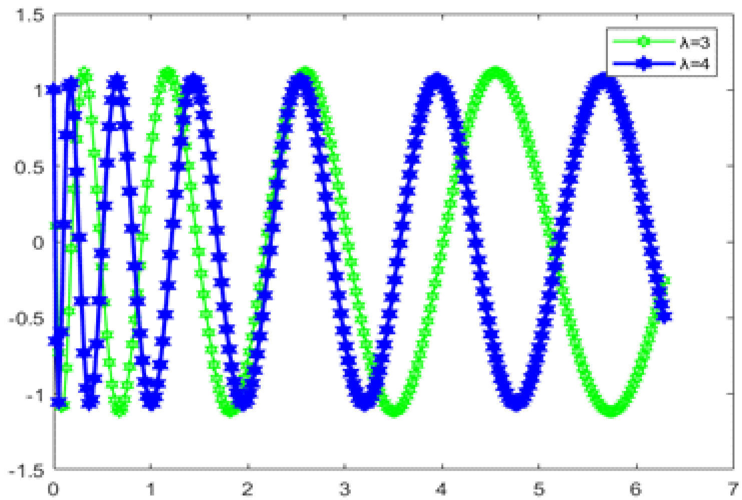

Fig. 2. Equation (15) image for two different λ values.

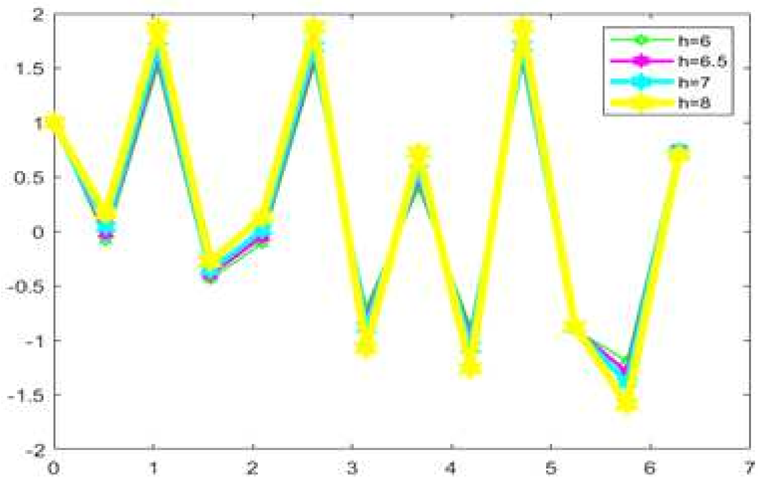

Fig. 3. Unique display of the solution for four kinds of h values.

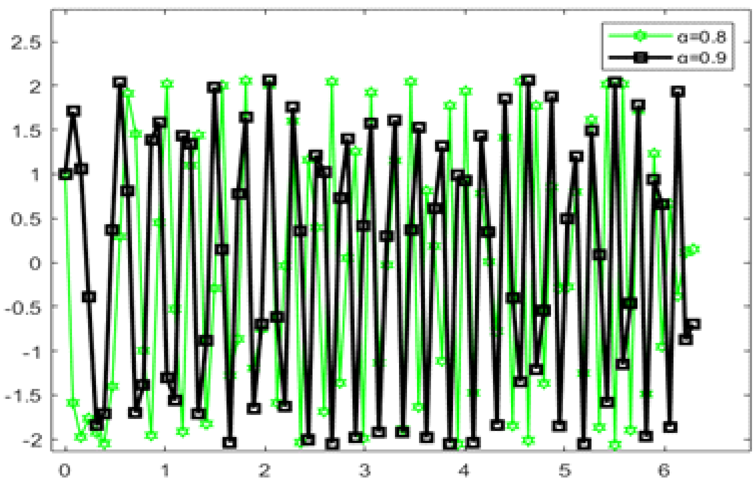

Fig. 4. Appearance of the solution as α approaches 1.

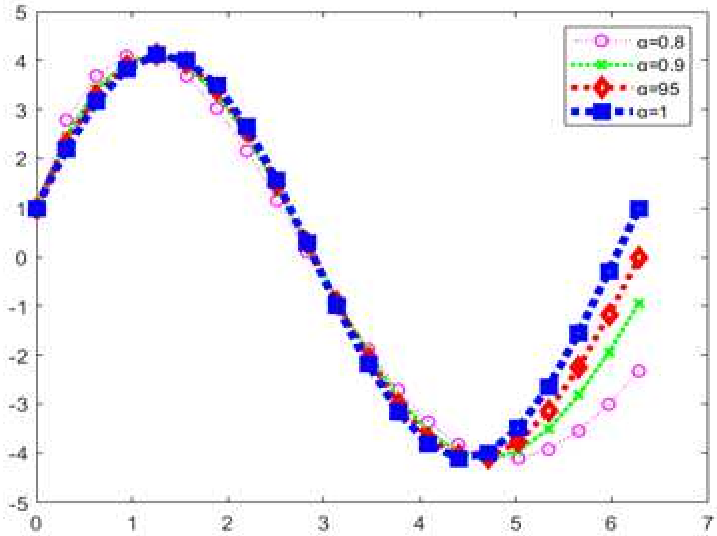

Fig. 5. View of the solution for several values of α.

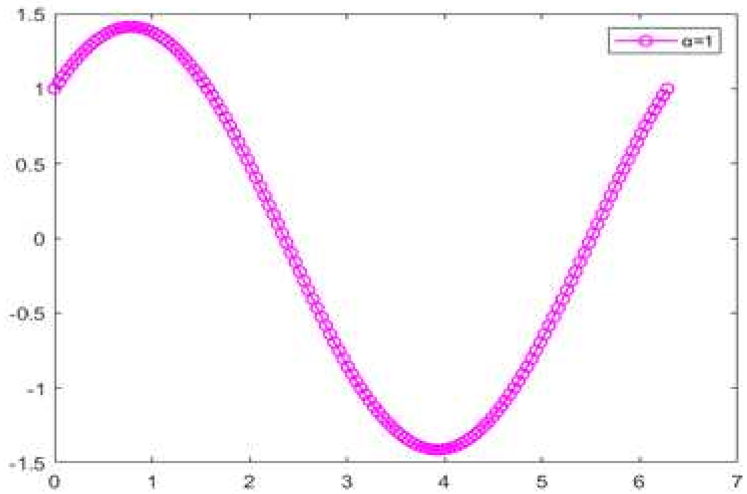

Fig. 6. Representation of solution in classical analysis.

The first figure is examined for diverse values of γ in equation (15). $\lambda = 2$, $\alpha = 0.2$, $h = 1$, $\gamma = 1$, $\gamma = 2$ and $\gamma = 3$ values by using the representation of the solution $\mathcal{M}$-Sturm Liouville problem for diffusion operator shape is appeared in detail. The second figure is followed the image of the solution $\mathcal{M}$-Sturm Liouville problem for diffusion operator in equation (15), taking into account the values of $\lambda = 3$, $\lambda = 4$, $\gamma = 2$, $\alpha = 0.5$ and $h = 1.5$. In the third figure, is obtained in the light of the values $\lambda = 5$, $\gamma = 3$, $\alpha = 0.6$,

$h = 6$, $h = 6.5$, $h = 7$ and $h = 8$. In the fourth figure, the $\mathcal{M}$-Sturm Liouville problem for diffusion operator figure is scrutinized by recording the $\lambda = 2.5$, $\gamma = 4$, $h = 4.5$, $\alpha = 0.8$ and $\alpha = 0.9$ in equation (15). In the fifth figure, the appearance of the $\mathcal{M}$-Sturm Liouville problem for the diffusion operator is observed by recording the dates $\lambda = 1$, $\gamma = 1$, $h = 4$, $\alpha = 0.8$, $\alpha = 0.9$, $\alpha = 0.95$ and $\alpha = 1$ in equation (15). In the sixth figure, there is recorded the figure of the $\mathcal{M}$-Sturm Liouville problem for diffusion operator representation of solution with the dates $\lambda = 1$, $\gamma = 1$, $h = 1$ and $\alpha = 1$ in equation (15). Also, sixth figure coincides with the representation of the solution in classical analysis for $\alpha = 1$.

5. Conclusions

In our study, is examined the $\mathcal{M}$-Sturm Liouville problem for diffusion operator, which is very important in mathematics. By revising the $\mathcal{M}$-Sturm Liouville problem for diffusion operator, which is a powerful derivative in fractional analysis, is obtained the representation of the solution with the help of $\mathcal{M}$-Laplace transform. The representation of this unique solution is observed in the light of various data through Matlab. The resulting solution representation has proven to be a more general version of classical analysis.

References

1. Acay B., Bas E., Abdeljawad Th. Non-local fractional calculus from different viewpoint generated by truncated M-derivative. *J. Comput. Appl. Math.*, 2020, **366** (112410), pp. 1-18.
2. Bas E. The Inverse nodal problem for the fractional diffusion equation. *Acta Scientiarum Technology*, 2015, **37** (2), pp. 251-257.
3. Bas E., Acay B. The direct spectral problem via local derivative including truncated Mittag-Leffler function. *Appl. Math. Comput.*, 2020, **367** (124787), pp 1-20.
4. Guseinov I.M., Nabiev I.M. The inverse spectral problem for pencils of differential operators. *Sb. Math.*, 2007, **198** (11), pp. 1579-1598.
5. Jarad F., Abdeljawad Th. Generalized fractional derivatives and Laplace transform. *Discrete Contin. Dyn. Syst. Ser. S*, 2020, **13** (3), pp. 709-722.
6. Jaulent M., Jean C. The inverse s-wave scattering problem for a class of potentials depending on energy. *Comm. Math. Phys.*, 1972, **28**, pp. 177-220.
7. Koyunbakan H., Panakhov E.S. Half-inverse problem for diffusion operators on the finite interval. *J. Math. Anal. Appl.*, 2007, **326** (2), pp. 1024-1030.
8. Miller K.S., Ross B. *An Introduction to the Fractional Calculus and Fractional Differential Equations.* Wiley, New York, 1993.
9. Podlubny I. *Fractional Differential Equations. Mathematics in Science and Enginering.* Academic Press, San Diego, 1999.
10. Sharma L.K., Luhanga P.V., Chimidza S. Potentials for the Klein-Gordon and Dirac equations. *Chiang Mai J. Sci.*, 2011, **38** (4), pp. 514-526.
11. Tuan V.K. Inverse problem for fractional diffusion equation. *Fract. Calc. Appl. Anal.*, 2011, **14** (1), pp. 31-55.

12. Vanterler J. da C. Sousa, Capelas E. de Oliveira. A new truncated M-fractional derivative type unifying some fractional derivative types with classical properties. *Int. J. Anal. Appl.*, 2018, **16** (1), pp. 83-96.
13. Wazwaz A.-M. *Partial Differential Equations: Methods and Applications.* CRC Press, 2002.
14. Yılmaz E. *Inverse Nodal Problem for Differential Operators.* PhD Thesis, Fırat University, 2012.