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MULTIPLE EIGENFUNCTION EXPANSION FOR ONE DIMENSIONAL SCHRÖDINGER EQUATION WITH PIECEWISE-CONSTANT COEFFICIENT

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In memory of M. G. Gasymov on his 85th birthday

Abstract. *In this work, direct scattering problems on the real axis for the one-dimensional Schrödinger equation with a piecewise constant coefficient is studied.*

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1. Introduction

Consider the differential operator L generated by the differential expression

$$l\left(\frac{d}{dx},\lambda\right)\equiv\frac{1}{\rho(x)}\left(-\frac{d^2}{dx^2}+2\lambda p(x)+q(x)\right)$$

in the space $L_2(-\infty,\infty)$.

Here λ is a spectral parameter $p(x)$, $q(x)$, and $\rho(x)$ functions satisfying the following conditions:

$$p(x)=\sum_{n=1}^{\infty}p_n e^{i\alpha_n x},\ \ \sum_{n=1}^{\infty}\alpha_n|p_n|<\infty;$$

$$q(x)=\sum_{n=1}^{\infty}q_n e^{i\alpha_n x},\ \ \sum_{n=1}^{\infty}|q_n|<\infty;$$

$$\rho(x)=\begin{cases}1 & \text{for} \quad x\geq 0,\\ \beta^2 & \text{for} \quad x<0.\end{cases}$$

Note that for the number set $G=\{\alpha_n\}$ the following conditions are satisfied:

1. $\alpha_1<\alpha_2<.....<\alpha_n<...,\quad \alpha_n\to\infty$;

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2. for any two numbers $\alpha_i, \alpha_j \in G$, $\alpha_i + \alpha_j \in G$.

The direct and inverse scattering problems for Schödinger operators with almost periodic potentials in the case $\rho(x) \equiv 1$ have been studied (see [3] and references therein).

We call the functions $p\left(x\right)$, $q\left(x\right)$ potentials of the equation

$$-y'' + 2\lambda p(x)y + q\left(x\right)y = \lambda^2 \rho\left(x\right)y \tag{1}$$

or operator L.

In this work, we will investigate the spectrum and obtain the eigenfunction expansion of the operator L by using the constructed particular solution equation (1). The eigenfunction expansion of the resolvent in terms of the continuous spectrum eigenfunction and the multiple expansions of arbitrary test functions were obtained. The formulae for expansion are obtained by summing the eigenfunctions of the corresponding differential equation.

2. Representation of Fundamental Solutions

In [2], it was proven, that the following equation

$$-y'' + 2\lambda p(x)y + q\left(x\right)y = \lambda^2 \rho\left(x\right)y$$

has particular solutions of the form

$$f_1^{\pm}\left(x,\lambda\right) = e^{\pm i\lambda x}\left(1 + \sum_{n=1}^{\infty} V_n^{\pm} e^{i\alpha_n x} + \sum_{n=1}^{\infty} \frac{1}{\alpha_n \pm 2\lambda} \sum_{s=n}^{\infty} V_{ns}^{\pm} e^{i\alpha_s x}\right), \quad for\ x > 0,$$

$$f_2^{\pm}\left(x,\lambda\right) = e^{\mp i\beta\lambda x}\left(1 + \sum_{n=1}^{\infty} V_n^{\pm} e^{i\alpha_n x} + \sum_{n=1}^{\infty} \frac{1}{\alpha_n \mp 2\beta\lambda} \sum_{s=n}^{\infty} V_{ns}^{\pm} e^{i\alpha_s x}\right), \quad for\ \ x < 0,$$

where the numbers $V_n^{\pm}$ and $V_{nn}^{\pm}$ $n < \alpha$, $n, \alpha \in \mathbb{N}$, are determined by the recurrent relations

$$\alpha_s^2 V_s^{\pm} + \alpha_s \sum_{n=1}^{s} V_{ns}^{\pm} + \sum_{\alpha_r + \alpha_k = \alpha_s}^{\alpha - 1} \left(q_r V_k^{\pm} \pm p_r \sum_{n=1}^{k} V_{nk}^{\pm}\right) + q_s = 0,$$

$$\alpha_s(\alpha_s - \alpha_n) V_{ns}^{\pm} + \sum_{\substack{\alpha_r + \alpha_k = \alpha_s \\ k \geq n}}^{\alpha - 1} \left(q_r \mp \alpha_n p_r\right) V_{nk}^{\pm} = 0,$$

$$\alpha_s V_s^{\pm} \pm \sum_{\alpha_r + \alpha_k = \alpha}^{\alpha - 1} p_r V_k^{\pm} \pm p_s = 0$$

and the series

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n} \sum_{s=n}^{\infty} \alpha_s \left|V_{ns}^{\pm}\right|,$$

$$\sum_{n=1}^{\infty} \alpha_n{}^2 \left| V_n^{\pm} \right|$$

are convergent.

The functions $f_1^+(x,\lambda), f_1^-(x,\lambda)$ and $f_2^+(x,\lambda), f_2^-(x,\lambda)$ are linearly independent for $\lambda \neq -\frac{n}{2}$, $\lambda \neq \frac{n}{2\beta}$ respectively and

$$\lim_{\dot{I}mx \to \infty} f_1^{\pm(\tau)}(x,\lambda)\, e^{\mp i\lambda x} = (\pm i\lambda x)^{\tau},\ \tau = 0,1,$$

$$\lim_{\dot{I}mx \to \infty} f_2^{\pm(\tau)}(x,\lambda)\, e^{\pm i\lambda\beta x} = (\mp i\lambda x)^{\tau},\ \tau = 0,1.$$

The functions $f_1^{\pm}(x,\lambda), f_2^{\pm}(x,\lambda)$ can be extended as a solution of equation (1) as follows:

- for $x > 0$

$$f_2^{\pm}(x,\lambda) = A^{\pm}(\lambda) f_1^{+}(x,\lambda) + B^{\pm}(\lambda) f_1^{-}(x,\lambda),$$

- for $x < 0$

$$f_1^{\pm}(x,\lambda) = C^{\pm}(\lambda) f_2^{+}(x,\lambda) + D^{\pm}(\lambda) f_2^{-}(x,\lambda),$$

where

$$A^{\pm}(\lambda) = \frac{1}{2i\lambda} W[f_2^{\pm}(x,\lambda), f_1^{-}(x,\lambda)],$$

$$B^{\pm}(\lambda) = \frac{1}{2i\lambda} W[f_2^{\pm}(x,\lambda), f_1^{+}(x,\lambda)],$$

$$C^{\pm}(\lambda) = \frac{1}{2i\lambda\beta} W[f_1^{\pm}(x,\lambda), f_2^{-}(x,\lambda)],$$

$$D^{\pm}(\lambda) = \frac{1}{2i\lambda\beta} W[f_1^{\pm}(x,\lambda), f_2^{+}(x,\lambda)],$$

if we denoted by $W[f,g] = f'g - fg'$.

It is proven [1] that the resolvent of the operator pencil L has the form

$$R_1(x,t,\lambda) = \frac{1}{W[f_1^+(x,\lambda), f_2^+(x,\lambda)]} \begin{cases} f_1^+(x,\lambda)\, f_2^+(t,\lambda), & t \leq x, \\ f_1^+(t,\lambda)\, f_2^+(x,\lambda), & t \geq x, \end{cases} \quad \mathrm{Im}\lambda > 0;$$

$$R_2(x,t,\lambda) = \frac{1}{W[f_1^-(x,\lambda), f_2^-(x,\lambda)]} \begin{cases} f_1^-(x,\lambda)\, f_2^-(t,\lambda), & t \leq x, \\ f_1^-(t,\lambda)\, f_2^-(x,\lambda), & t \geq x, \end{cases} \quad \mathrm{Im}\lambda < 0.$$

The spectra of the operator pencil L have a finite number of eigenvalues, the residual spectrum is empty, and the continuous spectrum fills the axis $\{Re\lambda = 0\}$ and may have a spectral singularity of the operator L accovding to Naimark [4, p.450] at the points $\pm\frac{n}{2},\ \pm\frac{n}{2\beta},\ n \in \mathbb{N}$.

3. Eigenfunction Expansion of the Resolvent in Terms of the Continuous Spectrum Eigenfunction

Let $\varGamma_0^+ \left(\varGamma_0^-\right)$ be a contour formed by segments

$$[0, \frac{1}{2} - \delta], [\frac{\alpha_n}{2} + \delta, \frac{(\alpha_n + 1)}{2} - \delta], [\frac{\alpha_n}{2\beta} + \delta, \frac{(\alpha_n + 1)}{2\beta} - \delta], \;\; n = 1, 2, ...,$$

and semicircles of radius δ with centres at points

$$\pm\frac{\alpha_n}{2}, \; \pm\frac{\alpha_n}{2\beta}, \; n \in \mathbb{N},$$

located in the upper (lower) half-plane. Let the numbers $\lambda_1, \lambda_2, \lambda_3 .... \lambda_n$ be an eigenvalue of the operator $L(\lambda)$.

The similar problem on a whole axis in the case $\rho(x) \equiv 1$, for classical Schrödinger equation were investigated in [5].

Theorem 1. *The resolvent operator $R(x,t,\lambda)$ of the operator pencil L admits expansion in terms of the continuous spectrum eigenfunctions*

$$\frac{1}{2i\pi}\int\limits_{\varGamma_N} \frac{R(x,t,\lambda)}{\lambda - z} d\lambda = R(x,t,z) + \sum_{i=1}^{n} res\left(R(x,t,\lambda)\right)|_{\lambda=\lambda_i} +$$

$$+\int\limits_{\varGamma_0^-} \frac{R_2(x,t,\lambda)}{\lambda - z} d\lambda - \int\limits_{\varGamma_0^+} \frac{R_1(x,t,\lambda)}{\lambda - z} d\lambda =$$

$$= R(x,t,z) + \sum_{i=1}^{n} res\left(R(x,t,\lambda)\right)|_{\lambda=\lambda_i} + \int\limits_{\varGamma_0^-} \frac{R_2(x,t,\lambda) - R_1(x,t,\lambda)}{\lambda - z} d\lambda +$$

$$+\sum_{n=1}^{\infty} res\left(R(x,t,\lambda)\right)|_{\lambda=\pm\frac{\alpha_n}{2}} + \sum_{n=1}^{\infty} res\left(R(x,t,\lambda)\right)|_{\lambda=\pm\frac{\alpha_n}{2\beta}}.$$

Here,

$$R_2(x,t,\lambda) - R_1(x,t,\lambda) =$$

$$= \frac{1}{2i\lambda\beta C^-(\lambda)} f_1^-(x,\lambda) f_2^-(t,\lambda) - \frac{1}{2i\lambda\beta D^+(\lambda) C^+(\lambda)} f_1^+(x,\lambda) f_1^+(t,\lambda) +$$

$$-\frac{1}{2i\lambda\beta C^+(\lambda)} f_1^+(x,\lambda) f_2^-(x,\lambda) = -\frac{1}{2i\lambda\beta D^+(\lambda) C^+(\lambda)} f_1^+(x,\lambda) f_1^+(t,\lambda) +$$

$$+\frac{1}{2i\lambda\beta C^+(\lambda) C^-(\lambda)} \left[C^+(\lambda) f_1^-(x,\lambda) f_2^-(t,\lambda) - C^-(\lambda) f_1^+(x,\lambda) f_2^-(t,\lambda)\right]$$

and

$$\lim_{\lambda \to \frac{\alpha_n}{2}} (\alpha_n - 2\lambda) R_1(x,t,\lambda) = \lim_{\lambda \to \frac{\alpha_n}{2}} (\alpha_n - 2\lambda) \frac{1}{W[f_1^+, f_2^+]} f_1^+(x,\lambda) f_2^+(t,\lambda) =$$

$$= \lim_{\lambda\to\frac{\alpha_n}{2}} (\alpha_n - 2\lambda) \frac{1}{W[f_1^+, f_2^+]} f_1^+ (x,\lambda) [\frac{W[f_2^+, f_1^-]}{2i\lambda} f_1^+ (t,\lambda) + \frac{W[f_1^+, f_2^+]}{2i\lambda} f_1^- (t,\lambda)] =$$

$$= \lim_{\lambda\to\frac{\alpha_n}{2}} \frac{(\alpha_n - 2\lambda)}{2i\lambda} [\frac{W[f_2^+, f_1^-]}{W[f_1^+, f_2^+]} f_1^+ (x,\lambda) f_1^+ (t,\lambda) + f_1^+ (x,\lambda) f_1^- (t,\lambda)] =$$

$$= \frac{i}{\alpha_n} [V_{nn}^+ f_1^+ \left(x, \frac{\alpha_n}{2}\right) f_1^+ \left(t, \frac{\alpha_n}{2}\right) + V_{nn}^+ f_1^+ \left(x, \frac{\alpha_n}{2}\right) f_1^+ \left(t, \frac{\alpha_n}{2}\right)] =$$

$$= \frac{2}{i\alpha_n} V_{nn}^+ f_1^+ \left(x, \frac{\alpha_n}{2}\right) f_1^+ \left(t, \frac{\alpha_n}{2}\right).$$

Analogously, we can show that

$$\lim_{\lambda\to\frac{\alpha_n}{2\beta}} (\alpha_n - 2\lambda\beta) R_1 (x,t,\lambda) = \lim_{\lambda\to\frac{\alpha_n}{2\beta}} (\alpha_n - 2\lambda\beta) \frac{1}{W[f_1^+, f_2^+]} f_1^+ (x,\lambda) f_2^+ (t,\lambda) =$$

$$= \lim_{\lambda\to\frac{\alpha_n}{2\beta}} \frac{(\alpha_n - 2\lambda\beta)}{2i\lambda\beta} [\frac{W[f_1^+, f_2^-]}{W[f_1^+, f_2^+]} f_2^+ (x,\lambda) f_2^+ (t,\lambda) + f_2^+ (x,\lambda) f_2^- (t,\lambda)] =$$

$$= \lim_{\lambda\to\frac{\alpha_n}{2\beta}} \frac{(\alpha_n - 2\lambda\beta)}{2i\lambda\beta} [\frac{W[f_2^+, f_1^-]}{W[f_1^+, \frac{f_{n2}^+(x)}{(\alpha_n - 2\lambda\beta)} + F_2^+ (x,\lambda)]} f_2^+ (x,\lambda) f_2^+ (t,\lambda) + f_2^+ (x,\lambda) f_2^- (t,\lambda)] =$$

$$= \lim_{\lambda\to\frac{\alpha_n}{2\beta}} \frac{(\alpha_n - 2\lambda\beta)}{2i\lambda\beta} [\frac{(\alpha_n - 2\lambda) W[f_2^+, f_1^-]}{W[f_1^+, f_{n2}^+ (x) + (\alpha_n - 2\lambda\beta) F_2^+ (x,\lambda)]} f_2^+ (x,\lambda) f_2^+ (t,\lambda) + f_2^+ (x,\lambda) f_2^- (t,\lambda)] =$$

$$= \lim_{\lambda\to\frac{\alpha_n}{2\beta}} \frac{1}{2i\lambda\beta} [\frac{W[f_2^+, f_1^-]}{W[f_{n1}^+ (x) + (\alpha_n - 2\lambda) F_1^+ (x,\lambda), f_2^+]} (\alpha_n - 2\lambda\beta) f_2^+ (x,\lambda) (\alpha_n - 2\lambda\beta) f_2^+ (t,\lambda) +$$

$$+ (\alpha_n - 2\lambda\beta) f_2^+ (x,\lambda) f_2^- (t,\lambda)] =$$

$$= -\frac{i}{\alpha_n\beta} [V_{nn}^- f_2^- \left(x, \frac{\alpha_n}{2\beta}\right) V_{nn}^- f_2^- \left(t, \frac{\alpha_n}{2\beta}\right) + V_{nn}^- f_2^- \left(x, \frac{\alpha_n}{2\beta}\right) f_2^- \left(t, \frac{\alpha_n}{2\beta}\right)].$$

4. Multiple Eigenfunction Expansion

Let $f_i (x)$, $i = 0, 1$, be arbitrary functions that are identically equal to zero in some neighbourhoods of infinite and 4 times differentiable. Consider the differential equation

$$-y'' + (2\lambda p (x) + q (x))y - \lambda^2 \rho (x) y = \lambda\varphi_0 (x) + \varphi_1 (x), \tag{2}$$

where

$$\varphi_0 (x) = f_0 (x), \quad \varphi_1 (x) = f_1 (x) + A_1 (x) \rho (x) f_0 (x).$$

Using the general theory of differential equations, the solution of equation (2) can be written as

$$y (x,\lambda) = R (x,t,\lambda) (\lambda\varphi_0 (x) + \varphi_1 (x)).$$

Theorem 2. *For a sufficiently large* $\lambda \in \bigcup_{\nu=0}^{3} S_\nu$*, we have the representations*

$$\frac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} \lambda^m y\left(x,\lambda\right)d\lambda = -\rho\left(x\right)f_m\left(x\right) + o\left(\varepsilon\right) + O(\frac{1}{\lambda^2}),\ m=0,1,$$

$$\frac{1}{2i\pi}\int\limits_{\Gamma_{N}} \lambda^m y\left(x,\lambda\right)d\lambda = -\rho\left(x\right)f_m\left(x\right) + O(\frac{1}{\lambda^2}),\ m=0,1.$$

Considering that

$$|R\left(x,t,\lambda\right)| \leq \frac{C}{|\lambda|}e^{-\tau|x-t|},$$

$$C = C\left(\lambda\right), \quad \tau = \min\{\mathrm{Im}\lambda, \mathrm{Re}\lambda\}, \quad \forall x,t \in \mathbb{R}$$

for $N \to \infty$ or for $|\lambda| \to \infty$, we obtain

$$\frac{1}{2i\pi}\int\limits_{\Gamma_N} \frac{R\left(x,t,\lambda\right)}{\lambda - z}d\lambda = 0.$$

Then, we have

$$R\left(x,t,z\right) = -\int\limits_{\Gamma_0^-} \frac{R_2\left(x,t,\lambda\right) - R_1\left(x,t,\lambda\right)}{\lambda - z}d\lambda -$$

$$-\sum_{n=1}^{\infty} res\left(R_1\left(x,t,\lambda\right)\right)|_{\lambda=\pm\frac{\alpha_n}{2}} - \sum_{n=1}^{\infty} res\left(R_1\left(x,t,\lambda\right)\right)|_{\lambda=\pm\frac{\alpha_n}{2\beta}} - \sum_{i=1}^{n} res\left(R\left(x,t,\lambda\right)\right)|_{\lambda=\lambda_i}.$$

Finally, let us denote by $G\left(x,t,\lambda_n\right)$ residues of the resolvent

$$R\left(x,t,\lambda\right) = \begin{cases} R_1\left(x,t,\lambda\right), & \mathrm{Im}\lambda > 0, \\ R_2\left(x,t,\lambda\right), & \mathrm{Im}\lambda > 0, \end{cases}$$

at the points $\lambda_1, \lambda_2, \lambda_3, ...\lambda_n$. The function $G\left(x,t,\lambda_n\right)$ is determined by the formula

$$G\left(x,t,\lambda_n\right) = \lim_{\lambda \to \lambda_n}\left(\lambda - \lambda_n\right)R\left(x,t,\lambda\right).$$

Therefore, for arbitrary elements $\psi\left(x\right)$ of the space $L_2\left(-\infty,\infty,\rho\left(x\right)\right)$, we have the following eigenfunction expansion:

$$\lim_{N\to\infty}\frac{1}{2i\pi}\int\limits_{\Gamma_N} \lambda^m y\left(x,\lambda\right)d\lambda = \frac{1}{2i\pi}\left(\int\limits_{\Gamma_0^+} \lambda^m y\left(x,\lambda\right)d\lambda + \int\limits_{\Gamma_0^-} \lambda^m y\left(x,\lambda\right)d\lambda\right) =$$

$$= \frac{1}{2i\pi}\int\limits_{\Gamma_0^+} \lambda^m \int\limits_{-\infty}^{\infty} R_1\left(x,\xi,\lambda\right)\left[\lambda\varphi_0\left(x\right) + \varphi_1\left(x\right)\right]d\xi d\lambda +$$

$$+\int\limits_{\Gamma_0^-} \lambda^m \int\limits_{-\infty}^{\infty} R_2\left(x,\xi,\lambda\right)\left[\lambda\varphi_0\left(x\right)+\varphi_1\left(x\right)\right]d\xi d\lambda +$$

$$+\sum_{n=1}^{\infty}\int\limits_{-\infty}^{\infty} \operatorname*{Res}_{\lambda=\lambda_n} R\left(x,\xi,\lambda\right)\left[\lambda_n\varphi_0\left(x\right)+\varphi_1\left(x\right)\right]d\xi =$$

$$=\frac{1}{2i\pi}\int\limits_{\Gamma_0^-} \lambda^m \int\limits_{-\infty}^{\infty}\left[R_2\left(x,\xi,\lambda\right)-R_1\left(x,\xi,\lambda\right)\right]\left[\lambda\varphi_0\left(x\right)+\varphi_1\left(x\right)\right]d\xi d\lambda +$$

$$+\sum_{n=1}^{\infty}\int\limits_{-\infty}^{\infty} \operatorname*{Res}_{\lambda=\pm\frac{\alpha_n}{2}} R_1\left(x,\xi,\lambda\right)\left[\pm\frac{\alpha_n}{2}\varphi_0\left(x\right)+\varphi_1\left(x\right)\right]d\xi +$$

$$+\sum_{n=1}^{\infty}\int\limits_{-\infty}^{\infty} \operatorname*{Res}_{\lambda=\pm\frac{\alpha_n}{2\beta}} R_1\left(x,\xi,\lambda\right)\left[\frac{\alpha_n}{2\beta}\varphi_0\left(x\right)+\varphi_1\left(x\right)\right]d\xi +$$

$$+\sum_{n=1}^{\infty}\int\limits_{-\infty}^{\infty} \operatorname*{Res}_{\lambda=\lambda_n} R\left(x,\xi,\lambda\right)\left[\lambda_n\varphi_0\left(x\right)+\varphi_1\left(x\right)\right]d\xi,$$

if we will take into account the fact that for sufficiently large λ

$$\frac{1}{2i\pi}\int\limits_{\Gamma_N}\lambda^m y\left(x,\lambda\right)d\lambda=-f_m+o\left(\varepsilon\right)+O\left(\frac{1}{\left|\lambda\right|^2}\right).$$

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