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ON THE INTEGRAL REPRESENTATIONS OF SOLUTIONS OF SOME MATRIX PENCILS OF THE STURM-LIOUVILLE EQUATION

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Abstract. *In the present work, we obtain some integral representations for special solutions, which play an important role in solving direct and inverse problems for second-order and fourth-order matrix Sturm-Liouville pencils. We also investigate some useful properties of the special solutions.*

Keywords: integral representation, direct problem, inverse problem, main integral equation, Sturm-Liouville equation, Fourier transformation, integral equation

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1. Introduction

Direct and inverse problems of spectral analysis for matrix Sturm-Liouville operators belong to the class of more complicated problems than the scalar one. Spectral analysis on the half-line for Sturm-Liouville operators in matrix form was carried out by Agranovich and Marchenko in [1], where the theory of the scalar case was carried over to the matrix case. For Sturm-Liouville operators in matrix form, direct and inverse problems are included in the studies of Carlson [6], [7], [8], with some special cases and restrictions. In these papers, the spectral properties of the matrix Sturm-Liouville operator with Drichlet and periodic boundary conditions are investigated, and a uniqueness theorem for the inverse problem is proved. Jodeit and Levitan [25] investigated direct and inverse

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problems for the matrix Sturm-Liouville operator with a symmetric potential. Direct and inverse problems on a finite interval for Sturm-Liouville operators in matrix form were investigated under various assumptions in [34], [11], [35], [19], [38]. However, these studies were carried out only for special cases and for the selfadjoint matrix Sturm-Liouville operator. In Yurko's work [39] for the matrix Sturm-Liouville problem on a finite interval, the method of spectral transformations was used to study the inverse problem. In this study, the problem is not self-adjoint, since the Sturm-Liouville operator has a complex-valued potential. In addition, in [30] some uniqueness theorems of the inverse problem for the matrix Sturm-Liouville equation are investigated, and in [10] necessary and sufficient conditions for the solvability of the inverse problem are obtained for the matrix Sturm-Liouville operator with a simple spectrum. For matrix potentials in the Sobolev space, the necessary and sufficient conditions for the solvability of the inverse problem from the spectral data are discussed in [32] . In recent years, direct and inverse problems for Sturm-Liouville operators in matrix form on a finite interval were investigated in [3], where the spectral properties of the matrix Sturm-Liouville operator were studied, an algorithm was developed for recovering the matrix potential by the method of spectral mappings, and necessary and sufficient conditions for the unique solvability of the inverse problem were obtained.

After studying the classical Sturm-Liouville operator, the Gelfand-Levitan-Faddeev-Marchenko's method [31], [28], [29], [18], [12], [13], [14] was generalized for the quadratic pencil of the Sturm-Liouville equation by Jaulent and Jean, Kaup [22], [23], [24], [27], [9]. Various inverse scattering and inverse spectral problems for the quadratic pencil of the Sturm-Liouville operator and other ordinary differential pencils were studied in [16], [17], [15], [36], [20], [21], [2], [37], [26], as well as in studies of many other authors. However, there are not many studies in the literature for a matrix Sturm-Liouville pencil [4], [5], [33]. In [4], a matrix Sturm-Liouville pencil with complex matrix coefficients and spectral parameters in boundary conditions is considered, and the uniqueness of the inverse problem with respect to the Weyl function is proved by the method of spectral mappings. In [5], the problem of determining matrix potentials based on spectral data is solved by transforming it into a system of linear equations in the Banach space. The fundamental system of solutions for the polynomial matrix pencil of the Sturm-Liouville equation was constructed in [33], where some useful integral representations for the solution were obtained.

In this study, we will obtain integral representations for special solutions, which play an important role in solving direct and inverse problems for second-order and fourth-order matrix Sturm-Liouville pencils, and we also investigate some useful properties of the special solutions.

2. On the Solutions of the Quadratic Pencil of the Matrix Sturm-Liouville Equation

In this section, the object of our research is the second-order matrix differential equation

$$-Y'' + (U(x) + \lambda Q(x)) = \lambda^2 Y, \quad 0 \leq x \leq \pi, \tag{1}$$

where

$$U(x) = \begin{pmatrix} q_0 & 0 & \ldots & 0 \\ q_1 & q_0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ q_{n-1} & q_{n-2} & \ldots & q_0 \end{pmatrix}$$

and

$$Q(x) = \begin{pmatrix} q_n & q_{n-1} & \ldots & q_1 \\ 0 & q_n & \ldots & q_2 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & q_n \end{pmatrix}$$

are $n \times n$ real matricies with compoments $q_j(x) \in L_2[0,\pi], j = \overline{0, n-1}$, and $q_n(x) \in W_1^2[0,\pi]$. Here $Y = (y_1, y_2, ..., y_n)^T$ is an unknown vector solution and λ is a spectral parameter. We consider the solutions of the matrix equation (1) satisfying the initial conditions

$$Y_j(0,\lambda) = V,\ Y_j^{'}(0,\lambda) = (-1)^{j+1} i\lambda V, \tag{2}$$

where $j = 1, 2$ and $V = (1, 0, ..., 0)^T$. The following integral equation is equivalent to the initial value problem (1), (2):

$$Y_j(x,\lambda) = e^{(-1)^{j+1} i\lambda x} V + \int_0^x \frac{\sin \lambda (x-t)}{\lambda} (U(t) + \lambda Q(t)) Y_j(t,\lambda) dt. \tag{3}$$

We search for the solution in the form

$$Y_j(x,\lambda) = e^{(-1)^{j+1} i\lambda x} f_j(x) V + \int_{-x}^x K_j(x,t) e^{(-1)^{j+1} i\lambda t} dt, \tag{4}$$

where the scalar function $f_j(x)$ and the vector kernel $K_j(x,t) = (K_{j1}(x,t), K_{j2}(x,t), ..., K_{jn}(x,t))^T$ will be defined later. Substituting the expression (4) for the $Y_j(x,\lambda)$ in the integral equation (3), we obtain

$$e^{(-1)^{j+1} i\lambda x} f_j(x) V + \int_{-x}^x K_j(x,t) e^{(-1)^{j+1} i\lambda t} dt =$$

$$= e^{(-1)^{j+1} i\lambda x} V + \int_0^x \frac{\sin \lambda(x-t)}{\lambda} (U(t) + \lambda Q(t)) e^{(-1)^{j+1} i\lambda t} f_j(t) V dt +$$

$$+ \int_0^x \frac{\sin \lambda(x-t)}{\lambda} (U)(t) + \lambda Q(t)) dt \int_{-t}^t K_j(t,s) e^{(-1)^{j+1} i\lambda s} ds. \tag{5}$$

Let us define a function $f_j(x)$ that satisfies the equation

$$f_j(x) V = V + \frac{(-1)^{j+1}}{2i} \int_0^x Q(t) f_j(t) V dt.$$

Consequently, since

$$Q(t)V=\begin{pmatrix} q_n & q_{n-1} & \dots & q_1 \\ 0 & q_n & \dots & q_2 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & q_n \end{pmatrix}\begin{pmatrix}1\\0\\ \vdots \\ 0\end{pmatrix}=\begin{pmatrix}q_n\\0\\ \vdots \\ 0\end{pmatrix},$$

then

$$f_j(x)=e^{\frac{(-1)^j}{2i}\int\limits_0^x q_n(t)dt}. \tag{6}$$

Hence, the equation (5) takes the form

$$\int\limits_{-x}^{x}K_j(x,t)e^{(-1)^{j+1}i\lambda t}dt=(-1)^{j+1}\int\limits_0^x\frac{e^{(-1)^{j+1}i\lambda x}-e^{(-1)^{j+1}i\lambda(2t-x)}}{2i\lambda}U(t)f_j(t)Vdt+$$

$$+\frac{(-1)^j}{2i}\int\limits_0^x e^{(-1)^{j+1}i\lambda(2t-x)}Q(t)f_j(t)Vdt+$$

$$+(-1)^{j+1}\int\limits_0^x\frac{e^{(-1)^{j+1}i\lambda(x-t)}-e^{(-1)^{j+1}i\lambda(t-x)}}{2i\lambda}(U(t)+$$

$$+\lambda Q(t))dt\int\limits_{-t}^{t}K_j(t,s)e^{(-1)^{j+1}i\lambda s}ds. \tag{7}$$

We can transform all integrals in the righthand side of equation (7) to the similar form of the lefthand side. Namely,

$$I_1=(-1)^{j+1}\int\limits_0^x\frac{e^{(-1)^{j+1}i\lambda x}-e^{(-1)^{j+1}i\lambda(2t-x)}}{2i\lambda}U(t)f_j(t)Vdt=$$

$$=\frac{1}{2}\int\limits_{-x}^{x}e^{(-1)^{j+i}\lambda t}dt\int\limits_0^{\frac{x+t}{2}}U(s)f_j(s)Vds,$$

$$I_2=\frac{(-1)^j}{2i}\int\limits_0^x e^{(-1)^{j+1}i\lambda(2t-x)}Q(t)f_j(t)Vdt=\frac{(-1)^j}{4i}\int\limits_{-x}^{x}e^{(-1)^{j+1}i\lambda t}Q(\frac{x+t}{2})f_j(\frac{x+t}{2})Vdt,$$

$$I_3=(-1)^{j+1}\int\limits_0^x\frac{e^{(-1)^{j+1}i\lambda(x-t)}-e^{(-1)^{j+1}i\lambda(t-x)}}{2i\lambda}Q(t)dt\int\limits_{-t}^{t}K_j(t,s)e^{(-1)^{j+1}i\lambda s}ds=$$

$$=\frac{1}{2}\int\limits_{-x}^{x}e^{(-1)^{j+1}i\lambda t}dt\int\limits_0^x U(s)ds\int\limits_{t-x+s}^{t+x-s}K_j(s,\xi)d\xi,$$

$$I_4=\frac{(-1)^{j+1}}{2i}\int_{-x}^{x}e^{(-1)^{j+1}i\lambda t}dt\int_{\frac{x-t}{2}}^{x}Q(s)K_j(s,t-x+s)ds-$$
$$-\frac{(-1)^{j+1}}{2i}\int_{-x}^{x}e^{(-1)^{j+1}i\lambda t}dt\int_{\frac{x+t}{2}}^{x}Q(s)K_j(s,t+x-s)ds.$$

Here we assume that $K_j(x,t)\equiv 0$ for $|t|>|x|$. Hence, after taking into the account the integrals I_1, I_2, I_3, and I_4 in the righthand side of the equation (7), we have

$$\int_{-x}^{x}K_j(x,t)e^{(-1)^{j+1}i\lambda t}dt=\int_{-x}^{x}e^{(-1)^{j+1}i\lambda t}dt\left\{\frac{1}{2}\int_{0}^{\frac{x+t}{2}}U(s)f_j(s)Vds+\right.$$
$$+\frac{(-1)^j}{4i}Q(\frac{x+t}{2})f_j(\frac{x+t}{2})V+\frac{1}{2}\int_0^x U(s)ds\int_{t-x+s}^{t+x-s}K_j(s,\xi)d\xi+$$
$$\left.+\frac{(-1)^{j+1}}{2i}\int_{\frac{x-t}{2}}^{x}Q(s)K_j(s,t-x+s)ds+\frac{(-1)^j}{2i}\int_{\frac{x+t}{2}}^{x}Q(s)K_j(s,t+x-s)ds\right\}.$$

Consequently, when $f_j(x)$ $(j=1,2)$ are defined by the formula (6), then the solution $Y_j(x,\lambda)$ of the matrix differential equation (1) has the form of (4) if and only if for each fixed x the kernel function $K_j(x,t)$ satisfies the integral equation

$$K_j(x,t)=\frac{1}{2}\int_0^{\frac{x+t}{2}}U(s)f_j(s)Vds+\frac{(-1)^{j+1}}{4}iQ(\frac{x+t}{2})f_j(\frac{x+t}{2})V+$$
$$+\int_0^{\frac{x+t}{2}}d\alpha\int_0^{\frac{x-t}{2}}U(\alpha+\beta)K_j(\alpha+\beta,\alpha-\beta)d\beta+$$
$$+\frac{(-1)^{j+1}}{2}i\int_0^{\frac{x-t}{2}}Q(\frac{x+t}{2}+\beta)K_j(\frac{x+t}{2}+\beta,\frac{x+t}{2}-\beta)d\beta-$$
$$-\frac{(-1)^{j+1}}{2}i\int_0^{\frac{x+t}{2}}Q(\alpha+\frac{x-t}{2})K_j(\alpha+\frac{x-t}{2},\alpha-\frac{x-t}{2})d\alpha,\ \ |t|\le|x|, \tag{8}$$

where $K_j(x,t)\equiv 0$ for $|t|>|x|$.

Now consider the integral equation (8) and define the successive approximations for this equation as follows:

$$K_j^{(0)}(x,t)=\frac{1}{2}\int_0^{\frac{x+t}{2}}U(s)f_j(s)V\,ds+\frac{(-1)^{j+1}}{4}iQ(\frac{x+t}{2})f_j(\frac{x+t}{2})V,$$

$$K_j^{(m+1)}(x,t)=\int_0^{\frac{x+t}{2}}dx\int_0^{\frac{x-t}{2}}U(\alpha+\beta)K_j^{(m)}(\alpha+\beta,\alpha-\beta)d\beta+$$

$$+\frac{(-1)^{j+1}}{2}i\int_0^{\frac{x-t}{2}}Q(\frac{x+t}{2}+\beta)K_j^{(m)}(\frac{x+t}{2}+\beta,\frac{x+t}{2}-\beta)d\beta-$$

$$-\frac{(-1)^{j+1}}{2}i\int_0^{\frac{x+t}{2}}Q(\alpha+\frac{x-t}{2})K_j^{(m)}(\alpha+\frac{x-t}{2},\alpha-\frac{x-t}{2})d\alpha,\ m=0,1,2...\ .$$

We define the matrix norm $\|A\|=\max\limits_j\sum\limits_k|a_{jk}|$ for the matrix $A=(a_{jk})$. Consequently, if $A=(a_1,a_2,...,a_n)^T$ then $\|A\|=\max\limits_{1\le k\le n}|a_k|$.

We have

$$\int_{-x}^{x}\left\|K_j^{(0)}(x,t)\right\|dt\le\frac{1}{2}\int_{-x}^{x}dt\left[\int_0^{\frac{x+t}{2}}\|U(s)\|\,ds+\frac{1}{2}\left\|Q(\frac{x+t}{2}\right\|\right]=$$

$$=\frac{1}{2}\int_0^x\left[2(x-s)\|U(s)\|+\|Q(s)\|\right]ds\le\int_0^x(x-s)\|U(s)\|\,ds+\int_0^x\|Q(s)\|\,ds,$$

$$\int_{-x}^{x}\left\|K_j^{(m+1)}(x,t)\right\|dt\le\frac{1}{2}\int_0^x(x-s)\|U(s)\|\,ds\int_{-s}^{s}\left\|K_j^{(m)}(s,\xi)\right\|d\xi+$$

$$+\int_0^x\|Q(s)\|\,ds\int_{-s}^{s}\left\|K_j^{(m)}\right\|d\xi\le\int_0^x((x-s)\|U(s)\|+\|Q(s)\|)ds\int_{-s}^{s}\left\|K_j^{(m)}(s,\xi)\right\|d\xi$$

which implies

$$\int_{-x}^{x}\left\|K_j^{(m+1)}(x,t)\right\|dt\le\frac{[\sigma(x)]^{m+1}}{(m+1)},\ m=0,1,2,...\ ,$$

where

$$\sigma(x)=\int_0^x\left[(x-s)\|U(s)\|+\|Q(s)\|\right]ds.$$

Hence the series

$$\sum_{m=0}^{\infty}\int_{-x}^{x} K_j^{(m)}(x,t)dt \tag{9}$$

is absolutely and uniformly convergent on the interval $(0,a)$ and we have

$$\int_{-x}^{x}\|K_j\,(x,t)\|\,dt \le e^{\sigma(x)}-1 \tag{10}$$

for the sum of the series (9). Consequently the integral equation (8) has a unique solution $K_j(x,\cdot)$ which satisfies the inequality (10).

Moreover, from the integral equation (8), we have

$$K_j(x,x)=\frac{1}{2}\int_0^x U(s)f_j(s)V ds+\frac{(-1)^{j+1}}{4}iQ(x)f_j(x)V+\frac{(-1)^j}{2}i\int_0^x Q(\alpha)K_j(\alpha,\alpha)d\alpha$$

and

$$K_j(x,-x)=\frac{(-1)^{j+1}}{4}iQ(0)V+\frac{(-1)^{j+1}}{2}i\int_0^x Q(\beta)Kj(\beta,\beta)d\beta.$$

Consequently, $K_j(x,x)$ and $K_j(x,-x)$ can be found as the solutions of the matrix differential equations

$$\frac{dK_j(x,x)}{dx}+\frac{(-1)^{j+1}}{2}iQ(x)K_j(x,x)=\frac{1}{2}U(x)f_j(x)V+\frac{(-1)^{j+1}}{4}i\frac{d}{dx}\left(Q(x)f_j(x)\right)V, \tag{11}$$

$$K_j(0,0)=\frac{(-1)^{j+1}}{4}iQ(0)V$$

and

$$\frac{dK_j(x,-x)}{dx}+\frac{(-1)^j}{2}iQ(x)K_j(x,-x)=0, \tag{12}$$

$$K_j(0,0)=\frac{(-1)^{j+1}}{4}iQ(0)V,$$

respectively. From the systems of differential equations (11) and (12), we obtain the following recurrence formulas for the $K_j(x,x)=(K_{j1}(x,x),...,K_{jn}(x,x))^T$ and $K_j(x,-x)=(K_{j1}(x,-x),...,K_{jn}(x,-x))^T$:

$$K_{jn}(x,x)f_j(x)=\frac{(-1)^{j+1}}{4}iq_n(0)+\frac{1}{2}\int_0^x q_{n-1}(t)f_j^2\,(t)\,dt, \tag{13}$$

$$K_{jm}(x,x)f_j(x)=\int_0^x \frac{1}{2}q_{n-1}(t)f_j^2(t)dt+$$

$$+\int_0^x \frac{(-1)^j}{2} i f_j(t) \sum_{\nu=m+1}^{n} q_{n-\nu+m}(t) K_{j\nu}(t,t)dt, \quad m=2,3,...,n-1 \tag{14}$$

$$K_{j1}(x,x)f_j(x)=\int_0^x \frac{1}{2} q_n(t) f_j^2(t)dt+\frac{(-1)^{j+1}}{4} i \int_0^x \left(q_n^{'}(t) f_j^2(t)+\frac{(-1)^{j+1}}{2} i q_n^2(t) f_j^2(t)\right) dt+$$

$$+\int_0^x \frac{(-1)^j}{2} i f_j(t) \sum_{\nu=2}^{n} q_{n-\nu+1}(t) K_{j\nu}(t)dt, \tag{15}$$

$$K_{jn}(x,-x)=f_j(x)\frac{(-1)^{j+1}}{4} i q_n(0), \tag{16}$$

$$K_{jm}(x,-x)=\frac{(-1)^{j+1}}{2} i \int_0^x \frac{f_j(x)}{f_j(t)} \sum_{\nu=m+1}^{n} q_{n-\nu+m}(t) K_{j\nu}(t,-t)dt \quad m=\overline{1,n-1}. \tag{17}$$

If we assume that $q_0(x)$ is differentiable and $q_j(x), j=1,2,...n-1$ are twice differentiable functions, i.e. $q_0(x)\in L_2[0,\pi], q_j''(x)\in L_2[0,\pi], 1,2,...n-1$, then coordinate functions of both partial derivatives $\frac{\partial^2 K_j(x,t)}{\partial x^2}$ and $\frac{\partial^2 K_j(x,t)}{\partial t^2}$ belong to the space $L_2[0,\pi]$ for each fixed $x\in[0,\pi]$. Moreover, $K_j(x,t)$ satisfies the following hyperbolic type partial differential equation

$$\frac{\partial^2 K_j(x,t)}{\partial x^2}-\frac{\partial^2 K_j(x,t)}{\partial t^2}=U(x)K_j(x,t)+(-1)^{j+1} iQ(x)\frac{\partial K_j(x,t)}{\partial t}. \tag{18}$$

Now we can formulate all results obtained above as the following theorem.

Theorem 1. *If $q_j(x)\in L_2[0,\pi]$ $(j=\overline{0,n-1})$, $q_n(x)\in W_2^1[0,\pi]$, then two solutions of the matrix equation (1) satisfying the conditions (2) can be represented in the integral form of (4), where the vector kernels $K_j(x,t)(j=1,2)$ are defined as the unique solution of the integral equalition (8) and $K_j(x,t)$ satisfies the inequality (10). Moreover, the vector kernels $K_j(x,t)$ satisfy equalitions (13)-(17). When we assume $q_0^{'}(x)\in L_2[0,\pi]$, $q^{''}(x)\in L_2[0,\pi]$ $(j=\overline{1,n})$, we have that $K_j(x,t)$ satisfy the partial differential equation (18).*

From Theorem 1 and the integral equations (8), we immediately have that there exist with respect to t the partial derivatives $\frac{\partial K_j(x,t)}{\partial x}$ and $\frac{\partial K_j(x,t)}{\partial t}$, so we have

$$\int_{-x}^{x}\left\|\frac{\partial K_j(x,t)}{\partial x}\right\| dt\leq h(x)e^{\sigma(x)}$$

and

$$\int_{-x}^{x}\left\|\frac{\partial K_j(x,t)}{\partial t}\right\| dt\leq h(x)e^{\sigma(x)}$$

for each $x\in[0,\pi]$, where $\sup\limits_{0\leq x\leq\pi}|h(x)|<+\infty$.

Note that from the integral representations (4) for the solutions $Y_1(x,\lambda)$ and $Y_2(x,\lambda)$ we immediately have

$$Y_j(x,\lambda) = e^{(-1)^{j+1}i\lambda x} f_j\left(x\right)V + \frac{(-1)^{j+1}}{i\lambda}K_j(x,x)e^{(-1)^{j+1}i\lambda x} -$$
$$-\frac{(-1)^{j+1}}{i\lambda}K_j(x,-x)e^{(-1)^{j}i\lambda x} + \frac{(-1)^{j+1}}{i\lambda}\int_{-x}^{x}\frac{\partial K_j(x,t)}{\partial t}e^{(-1)^{j}i\lambda t}dt.$$

3. On the Solutions of the Fourth Order Pencil of the Matrix Sturm-Liouville Equation

We consider the following matrix pencil of the Sturm-Liouville equation

$$-Y'' + (U(x) + \lambda V(x) + \lambda^2 Q(x))Y = \lambda^4 Y, \tag{19}$$

where $Y = (y_1, y_2, ..., y_n)^T$ is a solution, $U(x), V(x), Q(x)$ are $n \times n$ Hermitian matrix functions, the components of which are continuous complex valued functions on the interval $(0,a)$, $Q(x)$ is a lower triangular matrix function and λ is a complex parameter. Let $Y_k\left(x,\lambda\right)$ $(k=1,2)$ be vector solutions of the equation (19) satisfying the initial conditions

$$Y_k\left(0,\lambda\right) = I, Y_k'\left(0,\lambda\right) = \left(-1\right)^{k+1} i\lambda^2 I, \quad \lambda \in S_\nu, \tag{20}$$

where

$$S_\nu = \left\{\lambda : \frac{\nu\pi}{2} \le \arg\lambda \le \frac{\left(\nu+1\right)\pi}{2}\right\}, \quad \nu = 0,1,2,3,$$

and $I = (1,0,...,0)^T$.

Theorem 2. *Let $U(x), V(x), Q(x)$ be $n\times n$ Hermitian matrix functions, the components of which are continuous complex valued functions on the interval $(0,a)$, and $Q(x)$ is a lower triangular matrix function. Then the matrix equation (19) with initial conditions (20) has the solution $Y_k\left(x,\lambda\right)$ which is represented as*

$$Y_k\left(x,\lambda\right) = e^{(-1)^{k+1}i\lambda^2 x}\left\{R_k\left(x\right) + \int\limits_{\left[(-1)^{k+\nu}-1\right]\frac{x}{2}}^{+\infty} A_{l,\nu}\left(x,t\right)e^{(-1)^{\nu}2i\lambda^2 t}dt\right\}, \lambda \in S_\nu, \tag{21}$$

where $R_k\left(x\right)$ $A_{1,\nu}\left(x,t\right)$ and $A_{2,\nu}\left(x,t\right)$ are n-dimensional vector functions such that

$$\frac{d}{dx}R_k\left(x\right) = \frac{1}{2i}\left(-1\right)^{k+1}Q\left(x\right)R_k\left(x\right), R_k\left(0\right) = 0$$

and

$$A_{l,\nu}\left(x,.\right) \in L_1\left(\left[\left(-1\right)^{k+\nu}-1\right]\frac{x}{2};+\infty\right), l = k + \frac{1}{2}\left(\left(-1\right)^{k+1} + \left(-1\right)^{k+\nu}\right).$$

Proof. The matrix equation (19) can be written in coordinates as

$$-y_m'' + \sum_{j=1}^{n} \left(u_{mj} + \lambda v_{mj} + \lambda^2 q_{mj}\right) y_j = \lambda^4 y_m,\ m = 1, 2, ..., n, \tag{22}$$

where $U = (u_{mj})\,, V = (v_{mj})$ and $Q = (q_{mj})$ are given $n \times n$ matrices. The initial conditions (20) are in the form

$$y_m(0, \lambda) = 1, y_m'(0, \lambda) = (-1)^{k+1}\, i\lambda^2,\ k = 1, 2. \tag{23}$$

Therefore the problem (22), (23) is equivalent to the integral equation

$$y_{mk}(x, \lambda) = e^{(-1)^{k+1} i\lambda^2 x} +$$

$$+ \int_0^x \frac{\sin \lambda^2 (x - t)}{\lambda^2} \sum_{j=1}^{n} \left(u_{mj}(t) + \lambda v_{mj}(t) + \lambda^2 q_{mj}(t)\right) y_{jk}(t, \lambda) dt. \tag{24}$$

We seek the solution of (24) in the form

$$y_{mk}(x, \lambda) = e^{(-1)^{k+1} i\lambda^2 x} \left[R_{mk}\,(x) + z_{mk}(x, \lambda)\right], \tag{25}$$

where $z_{mk}(x, \lambda)$ is an unknown scalar function and the scalar functions $R_{mk}\,(x)$ are defined as the solution of the system of integral equations

$$R_{mk}\,(x) = 1 + (-1)^{k+1}\,(2i)^{-1} \int_0^x \sum_{j=1}^{m} q_{mj}(t) R_{jk}\,(t)\, dt. \tag{26}$$

After substituting (25) in the equation (24) we have

$$z_{mk}(x, \lambda) = (-1)^{k+1} \int_0^x \frac{1 - e^{2i(-1)^{k+1}\lambda^2 (t-x)}}{2i\lambda^2} \sum_{j=1}^{n} \left(u_{mj}(t) + \lambda v_{mj}(t)\right) R_{jk}(t) dt +$$

$$+ (-1)^k\,(2i)^{-1} \int_0^x e^{2i(-1)^{k+1}\lambda^2 (t-x)} \sum_{j=1}^{n} q_{mj}(t) R_{jk}(t) dt +$$

$$+ (-1)^{k+1} \int_0^x \frac{1 - e^{2i(-1)^{k+1}\lambda^2 (t-x)}}{2i\lambda^2} \sum_{j=1}^{n} \left(u_{mj}(t) + \lambda v_{mj}(t) + \lambda^2 q_{mj}(t)\right) z_{jk}(t, \lambda) dt. \tag{27}$$

If we define the vector function $R_k(x) = \left(R_{1k}(x), R_{2k}(x), ..., R_{nk}(x)\right)^T$, then from equations (26) we obtain

$$\frac{d}{dx} R_k(x) = (-1)^{k+1}\,(2i)^{-1}\, Q(x) R_k(x),\ R_k(0) = 0, \quad k = 1, 2.$$

If we set $Z_k(x) = (z_{1k}(x), z_{2k}(x), ...z_{nk}(x))^T$, then the system of equations (27) is written as

$$Z_k(x,\lambda) = (-1)^{k+1} \int_0^x \frac{1 - e^{2i(-1)^{k+1}\lambda^2(t-x)}}{2i\lambda^2} (U(t) + \lambda V(t)) R_k(t)dt+$$

$$+ (-1)^k (2i)^{-1} \int_0^x e^{2i(-1)^{k+1}\lambda^2(t-x)} Q(t) R_k(t) dt+$$

$$+ (-1)^{k+1} \int_0^x \frac{1 - e^{2i(-1)^{k+1}\lambda^2(t-x)}}{2i\lambda^2} \left(U(t) + \lambda V(t) + \lambda^2 Q(t)\right) Z_k(t,\lambda)dt. \tag{28}$$

We seek the solution of the integral equation (28) in the form

$$Z_k(x,\lambda) = \int_{\left[(-1)^{k+\nu}-1\right]\frac{x}{2}}^{+\infty} A_{l,\nu}(x,t)\, e^{(-1)^\nu 2i\lambda^2 t} dt, \ \lambda \in S_\nu, \tag{29}$$

where the kernel functions $A_{l,\nu}(x,t)$ $(l = k + \frac{1}{2}\left((-1)^{k+1} + (-1)^{k+\nu}\right),\ k = 1,2;\ \nu = 0,1,2,3)$ will be defined later. Substituting (29) in the system (28), we have

$$\int_{\left[(-1)^{l+\nu}-1\right]\frac{x}{2}}^{+\infty} A_{l,\nu}(x,t)\, e^{(-1)^\nu 2i\lambda^2 t} dt =$$

$$= (-1)^{k+1} \int_0^x \frac{1 - e^{2i(-1)^{k+1}\lambda^2(t-x)}}{2i\lambda^2} (U(t) + \lambda V(t)) R_k(t)dt+$$

$$+ (-1)^k (2i)^{-1} \int_0^x e^{2i(-1)^{k+1}\lambda^2(t-x)} Q(t) R_k(t) dt+$$

$$+ (-1)^{k+1} \int_0^x \frac{1 - e^{2i(-1)^{k+1}\lambda^2(t-x)}}{2i\lambda^2} \left(U(t) + \lambda V(t) + \lambda^2 Q(t)\right) dt \times$$

$$\times \int_{\left[(-1)^{l+\nu}-1\right]\frac{t}{2}}^{+\infty} A_{l,\nu}(t,\xi)\, e^{(-1)^\nu 2i\lambda^2 \xi} d\xi. \tag{30}$$

Now by using the formulas (see [28])

$$\frac{e^{(-1)^\nu 2i\lambda^2 x}}{2i\lambda} = (-1)^{\nu+1} \frac{\gamma_1^{(\nu)}}{\sqrt{\pi}} \int_x^\infty (s-x)^{-\frac{1}{2}} e^{(-1)^\nu 2i\lambda^2 t} dt, \quad \lambda \in S_\nu, \tag{31}$$

and

$$\frac{e^{(-1)^{\nu}2i\lambda^2x}-1}{2i\lambda^2}=(-1)^{\nu}\int\limits_0^x e^{(-1)^{\nu}2i\lambda^2t}dt$$

we convert the right hand side of the equation (30) similar to the left one. Here $\gamma_1^{(\nu)}=2^{-\frac{1}{2}}e^{\frac{i\pi}{4}(2\nu+1)},\nu=0,1,2,3.$

If we assume $k+\nu$ to be an odd number, after some integral order changings in the righthand side of the equation (30), we have

$$\int\limits_{-x}^{+\infty}A_{1,\nu}\left(x,t\right)e^{(-1)^{\nu}2i\lambda^2t}dt=$$

$$=\int\limits_{-x}^{0}e^{(-1)^{\nu}2i\lambda^2t}dt\left\{\int\limits_0^{x+t}U(s)R_k(s)ds+\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_0^{x+t}\left(x+t-s\right)^{-\frac{1}{2}}V\left(s\right)R_k(s)ds+\right.$$

$$+\frac{(-1)^{\nu+1}}{2i}Q(x+t)R_k(x+t)+\int\limits_0^x U\left(s\right)ds\int\limits_{-s}^{x+t-s}A_{1,\nu}\left(s,\xi\right)d\xi-$$

$$-\int\limits_{-t}^{x}U\left(s\right)ds\int\limits_{-s}^{t}A_{1,\nu}\left(s,\xi\right)d\xi+\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_0^xV\left(s\right)ds\int\limits_{-s}^{x+t-s}\left(x+t-s-\xi\right)^{-\frac{1}{2}}A_{1,\nu}\left(s,\xi\right)d\xi-$$

$$-\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_{-t}^{x}V\left(s\right)ds\int\limits_{-s}^{t}\left(t-\xi\right)^{-\frac{1}{2}}A_{1,\nu}\left(s,\xi\right)d\xi+\frac{(-1)^{\nu}}{2i}\int\limits_{-t}^{x}Q(s)A_{1,\nu}\left(s,t\right)ds-$$

$$\left.-\frac{(-1)^{\nu}}{2i}\int\limits_0^xQ(s)A_{1,\nu}\left(s,t-s+x\right)ds\right\}+$$

$$+\int\limits_0^{+\infty}e^{(-1)^{\nu}2i\lambda^2t}dt\left\{\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_0^x\left[\left(x+t-s\right)^{-\frac{1}{2}}-t^{-\frac{1}{2}}\right]V\left(s\right)R_k(s)ds+\right.$$

$$-\int\limits_0^xU\left(s\right)ds\int\limits_{-s}^{x+t-s}A_{1,\nu}\left(s,\xi\right)d\xi-\int\limits_0^xU\left(s\right)ds\int\limits_{-s}^{t}A_{1,\nu}\left(s,\xi\right)d\xi-$$

$$-\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_0^xV\left(s\right)ds\int\limits_{-s}^{t}\left(t-\xi\right)^{-\frac{1}{2}}A_{1,\nu}\left(s,\xi\right)d\xi+$$

$$+\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_0^xV(s)ds\int\limits_{-s}^{x+t-s}\left(x+t-s-\xi\right)^{-\frac{1}{2}}A_{1,v}\left(s,\xi\right)d\xi+$$

$$+\frac{(-1)^{\nu}}{2i}\int_0^x Q(s)A_{1,\nu}(s,\xi)\,ds-\frac{(-1)^{\nu}}{2i}\int_0^x Q(s)A_{1,\nu}(s,t-s+x)\,ds\Bigg\}.$$

Consequently, if the integral equation

$$A_{1,\nu}(x,t)=\begin{cases}\int_0^{x+t}U(s)R_k(s)ds+\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int_0^{x+t}(x+t-s)^{-\frac{1}{2}}V(s)R_k(s)ds+\\ +\frac{(-1)^{\nu+1}}{2i}Q(x+t)R_k(x+t)+\int_0^x U(s)\,ds\int_{-s}^{x+t-s}A_{1,\nu}(s,\xi)\,d\xi-\\ -\int_{-t}^{x}U(s)\,ds\int_{-s}^{t}A_{1,\nu}(s,\xi)\,d\xi+\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int_0^x V(s)\,ds\times\\ \times\int_{-s}^{x+t-s}(x+t-s-\xi)^{-\frac{1}{2}}A_{1,\nu}(s,\xi)\,d\xi-\\ -\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int_0^x V(s)ds\int_{-s}^{x+t-s}(x+t-s-\xi)^{-\frac{1}{2}}A_{1,v}(s,\xi)\,d\xi+\\ +\frac{(-1)^{\nu}}{2i}\int_0^x Q(s)A_{1,\nu}(s,t)\,ds-\\ -\frac{(-1)^{\nu}}{2i}\int_0^x Q(s)A_{1,\nu}(s,t-s+x)\,ds,\quad \text{if}\quad -x\le t<0,\\ \\ \frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int_0^x\left[(x+t-s)^{-\frac{1}{2}}-t^{-\frac{1}{2}}\right]V(s)R_k(s)ds+\\ +\int_0^x U(s)\,ds\int_{-s}^{x+t-s}A_{1,\nu}(s,\xi)\,d\xi-\int_0^x U(s)\,ds\int_{-s}^{t}A_{1,\nu}(s,\xi)\,d\xi-\\ -\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int_0^x V(s)\,ds\int_{-s}^{t}(t-\xi)^{-\frac{1}{2}}A_{1,\nu}(s,\xi)\,d\xi+\\ +\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int_0^x V(s)ds\int_{-s}^{x+t-s}(x+t-s-\xi)^{-\frac{1}{2}}A_{1,v}(s,\xi)\,d\xi+\\ +\frac{(-1)^{\nu}}{2i}\int_0^x Q(s)A_{1,\nu}(s,\xi)\,ds-\\ -\frac{(-1)^{\nu}}{2i}\int_0^x Q(s)A_{1,\nu}(s,t-s+x)\,ds,\quad \text{if}\quad t>0,\end{cases}\tag{32}$$

is satisfied for each $x\in[0,\pi]$, then the vector function $Y_k(x,\lambda)$ constructed by the formula

$$Y_k(x,\lambda)=e^{(-1)^{k+1}i\lambda^2x}\left(R_k(x)+\int_{-x}^{+\infty}A_{1,\nu}(x,t)\,e^{(-1)^{\nu}2i\lambda^2t}dt\right)$$

is the solution of the integral equation (23) for all $\lambda\in S_\nu$, where $k+\nu$ $\left(k=1,2;\ \nu=\overline{0,3}\right)$ is an odd number. Similarly we obtain that if $k+\nu$ $\left(k=1,2;\ \nu=\overline{0,3}\right)$ is an even number and the equation

$$A_{2,\nu}(x,t)=\begin{cases} \int\limits_0^{x-t} U(s)R_k(s)ds+\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_0^x t^{-\frac{1}{2}}V(s)\,R_k(s)ds- \\ -\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_{x-t}^{x}(x+t-s)^{-\frac{1}{2}}V(s)\,R_k(s)ds+ \\ +\frac{(-1)^\nu}{2i}Q(x-t)R_k(x-t)+\int\limits_0^x U(s)\,ds\int\limits_0^t A_{2,\nu}(s,\xi)\,d\xi- \\ -\int\limits_{x-t}^{x}U(s)\,ds\int\limits_0^{s-x+t}A_{2,\nu}(s,\xi)\,d\xi+ \\ +\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_0^x V(s)\,ds\int\limits_0^t (t-\xi)^{-\frac{1}{2}}A_{2,\nu}(s,\xi)\,d\xi- \\ -\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_{x-t}^{x}V(s)ds\int\limits_0^{s-x+t}(s-x+t-\xi)^{-\frac{1}{2}}A_{2,v}(s,\xi)\,d\xi- \\ -\frac{(-1)^\nu}{2i}\int\limits_0^x Q(s)A_{2,\nu}(s,t)\,ds+ \\ +\frac{(-1)^\nu}{2i}\int\limits_{x-t}^{x}Q(s)A_{2,\nu}(s,t-x+s)\,ds, \quad \text{if} \quad 0<t\le x, \\ \\ \frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_0^x\left[t^{-\frac{1}{2}}-(t-x+s)^{-\frac{1}{2}}\right]V(s)\,R_k(s)ds+ \\ +\int\limits_0^x U(s)\,ds\int\limits_0^t A_{2,\nu}(s,\xi)\,d\xi+\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_0^x V(s)\,ds\times \\ \times\int\limits_{t-x+s}^{t}(t-\xi)^{-\frac{1}{2}}A_{2,\nu}(s,\xi)\,d\xi+ \\ +\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_0^x V(s)ds\int\limits_0^{t-x+s}\left[(t-\xi)^{-\frac{1}{2}}-(t-x+s-\xi)^{-\frac{1}{2}}\right]\times \\ \times A_{2v}(s,\xi)\,d\xi-\frac{(-1)^\nu}{2i}\int\limits_0^x Q(s)A_{2,\nu}(s,t)\,ds+ \\ +\frac{(-1)^\nu}{2i}\int\limits_0^x Q(s)A_{2,\nu}(s,t-x+s)\,ds, \quad \text{if} \quad t\ge x, \end{cases} \tag{33}$$

is satisfied for each $x\in[0,\pi]$, then the vector function $Y_k(x,\lambda)$ constructed by the formula

$$Y_k(x,\lambda)=e^{(-1)^{k+1}i\lambda^2x}\left(R_k(x)+\int\limits_0^{+\infty}A_{2,\nu}(x,t)\,e^{(-1)^\nu 2i\lambda^2t}dt\right)$$

is the solution of the integral equation (23) for all $\lambda\in S_\nu$.

Now we use the method of successive approximations to prove the existence of the unique integrable solutions of the integral equations (32) and (33). We will prove it only for the equation (33). The statement can be proven for the equation (32) by the same way. We define for the equation (33) the successive aproximations $A_{2,\nu}^{(p)}(x,t)\,,p=0,1,2,...,$

by the following formulas:

$$A_{2,\nu}^{(0)}(x,t)=\begin{cases}\int\limits_0^{x-t}U(s)R_k(s)ds+\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_0^x t^{-\frac{1}{2}}V(s)R_k(s)ds-\\ -\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_{x-t}^x(x+t-s)^{-\frac{1}{2}}V(s)R_k(s)ds+\\ +\frac{(-1)^\nu}{2i}Q(x-t)R_k(x-t),\quad \text{if}\quad 0<t\le x,\\ \frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_0^x\left[t^{-\frac{1}{2}}-(t-x+s)^{-\frac{1}{2}}\right]V(s)R_k(s)ds,\quad \text{if}\quad t\ge x,\end{cases}\tag{34}$$

$$A_{2,\nu}^{(p)}(x,t)=\begin{cases}\int\limits_0^x U(s)\,ds\int\limits_0^t A_{2,\nu}^{(p-1)}(s,\xi)\,d\xi-\int\limits_{x-t}^x U(s)\,ds\int\limits_0^{s-x+t}A_{2,\nu}^{(p-1)}(s,\xi)\,d\xi+\\ +\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_0^x V(s)\,ds\int\limits_0^t(t-\xi)^{-\frac{1}{2}}A_{2,\nu}^{(p-1)}(s,\xi)\,d\xi-\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_{x-t}^x V(s)ds\times\\ \times\int\limits_0^{s-x+t}(s-x+t-\xi)^{-\frac{1}{2}}A_{2,v}^{(p-1)}(s,\xi)\,d\xi-\\ -\frac{(-1)^\nu}{2i}\int\limits_0^x Q(s)A_{2,\nu}^{(p-1)}(s,t)\,ds+\\ +\frac{(-1)^\nu}{2i}\int\limits_{x-t}^x Q(s)A_{2,\nu}^{(p-1)}(s,t-x+s)\,ds,\quad \text{if}\quad 0<t\le x,\\ \int\limits_0^x U(s)\,ds\int\limits_0^t A_{2,\nu}^{(p-1)}(s,\xi)\,d\xi+\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_0^x V(s)\,ds\times\\ \times\int\limits_{t-x+s}^t(t-\xi)^{-\frac{1}{2}}A_{2,\nu}^{(p-1)}(s,\xi)\,d\xi+\\ +\frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}}\int\limits_0^x V(s)ds\int\limits_0^{t-x+s}\left[(t-\xi)^{-\frac{1}{2}}-(t-x+s-\xi)^{-\frac{1}{2}}\right]\times\\ \times A_{2v}^{(p-1)}(s,\xi)\,d\xi-\frac{(-1)^\nu}{2i}\int\limits_0^x Q(s)A_{2,\nu}^{(p-1)}(s,t)\,ds+\\ +\frac{(-1)^\nu}{2i}\int\limits_0^x Q(s)A_{2,\nu}^{(p-1)}(s,t-x+s)\,ds,\quad \text{if}\quad t\ge x.\end{cases}\tag{35}$$

From the equation (34), we easily have

$$\int\limits_0^{+\infty}\left\|A_{2,\nu}^{(0)}\right\|(x,t)\,dt\le$$

$$\le C\int\limits_0^x\left[(x-s)\|U\|(s)+(x-s)^{\frac{1}{2}}\|V\|(s)+\|Q\|(s)\right]\|R_k\|(s)ds\tag{36}$$

for some constant $C>0$. Here $\|\cdot\|$ is a matrix norm defined as

$$\|A\|=\max_{1\le i\le n}\sum_{j=0}^n|a_{ij}|$$

for $n \times n$ matrix $A = (a_{ij})$. From the equation (35), we analogously obtain

$$\int_0^{+\infty} \left\| A_{2,\nu}^{(p)} \right\| (x,t)\, dt \le \int_0^x \left[(x-s) \|U\| (s) + (x-s)^{\frac{1}{2}} \|V\| (s) + \|Q\| (s) \right] ds \times$$

$$\times \int_0^{+\infty} \left\| A_{2,m}^{(p-1)} \right\| (s,t)\, dt, \ p = 1, 2, \dots .$$

By induction, we have

$$\int_0^{+\infty} \left\| A_{2,\nu}^{(p)} \right\| (x,t)\, dt \le C \max_{0 \le x \le a} \|R_k\| (x) \left[\sigma(x) \right]^{p+1}, \quad p = 0, 1, 2, \dots .$$

Here

$$\sigma(x) = \int_0^x \left[(x-s) \|U\| (s) + (x-s)^{\frac{1}{2}} \|V\| (s) + \|Q\| (s) \right] ds.$$

Therefore the series

$$\sum_{p=0}^{\infty} \int_0^{+\infty} A_{2,\nu}^{(p)} (x,t)\, dt$$

converges absolutely and uniformly with respect to x . Consequently the integral equation (33) has a unique integrable solution and

$$\int_0^{+\infty} \|A_{2,\nu}\| (x,t)\, dt \le C_1 e^{\sigma(x)} - 1$$

is satisfied for each $x \in [0, a]$ where $C_1 > 0$ is a constant.

By the same way we prove that Eq. (32) has the unique solution $A_{1,\nu}(x,t)$ for which the inequality

$$\int_{-x}^{+\infty} \|A_{1,\nu}\| (x,t)\, dt \le C_2 e^{\sigma(x)} - 1$$

is satisfied for some $C_2 > 0$ .Theorem is proved. ◀

Theorem 2 implies the following result:

Corollary 1. *Under hypothesis of Theorem 2 the problem (19)-(23) has a unique continuous vector solution which is represented in the form of (21) and this vector solution is analytic in* λ*.*

Proof. We will show this only analticity; other statements are direct results of Theorem 2. Let $\in S_{v+1} \cap S_v$, $\nu = 0,1,2,3$, i.e. $\arg\lambda = \frac{(\nu+1)\pi}{2}$. It is easy to see that in this case the system of equations (24) has the unique continuous solution $Y = (y_1, y_2, ..., y_n)^T$; on the other hand, by Theorem 2, this solution can be expressed by either (21) or

$$y_j(x,\lambda) = e^{(-1)^{j+1} i\lambda^2 x}\left(R_j(x) + \int\limits_{\left[(-1)^{j+\nu+1}-1\right]\frac{x}{2}}^{+\infty} A_{l,\nu+1}(x,t)\, e^{(-1)^{\nu+1}2i\lambda^2 t}dt\right). \qquad (21')$$

Therefore (21) and (21′) coincide for $\arg\lambda = \frac{(\nu+1)\pi}{n}$. Since the solutions (21) and (21′) are analytic for $\lambda \in S_\nu$ and $\lambda \in S$, respectively, we have that the solution of the problem (19) -(20) is analytic on the complex plane.

Theorem 3. *If $U(x) \in \mathbf{C}[0,a], V(x), Q(x) \in \mathbf{C}^1[0,a]$, then the vector functions*

$$\frac{\partial}{\partial x} A_{k,\nu}(x,.) \;\; (k = 1,2;\;\; \nu = 0,1,2,3)$$

are integrable on $\left(\left[(-1)^k - 1\right]\frac{x}{2}; +\infty\right)$ *and the inequality*

$$\int\limits_{\left[(-1)^k-1\right]\frac{x}{2}}^{+\infty} \left|\frac{\partial}{\partial x} A_{k,\nu}(x,t)\right| dt < +\infty$$

holds for each $x \in [0,\pi]$.

Proof. Consider the formulas (34) and (35). Formal differentation of the series (36) with respect to x gives

$$\sum_{p=0}^{\infty} \int\limits_0^{+\infty} \frac{\partial}{\partial x} A_{2,\nu}^{(p)}(x,t)\, dt. \qquad (37)$$

Show that the series (37) is absolutely and uniformly convergent. Rewrite Eq. (36) as

$$A_{2,\nu}^{(0)}(x,t) = \begin{cases} \int\limits_0^{x-t} U(s)R_k(s)ds + \frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}} \int\limits_t^x t^{-\frac{1}{2}} V(x-s) R_k(x-s)ds + \\ \quad + \frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}} \int\limits_0^t \left((t-s)^{-\frac{1}{2}} - t^{-\frac{1}{2}}\right) V(x-s) R_k(x-s)ds + \\ \quad + \frac{(-1)^\nu}{2i} Q(x-t) R_k(x-t), \quad \text{if} \;\; 0 < t \le x, \\ \frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}} \int\limits_0^x \left[t^{-\frac{1}{2}} - (t-s)^{-\frac{1}{2}}\right] V(x-s) R_k(x-s)ds, \quad \text{if} \;\; t \ge x. \end{cases}$$

If we compute the derivative in x, we have

$$D_x A_{2,\nu}^{(0)}(x,t) = \begin{cases} U(x-t)R_k(x-t) + \frac{(-1)^\nu}{2i}\frac{d}{dx}\left[Q(x-t)R_k(x-t)\ \right] + \\ + \frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}} \int\limits_t^x t^{-\frac{1}{2}} \frac{d}{dx}\left[V(x-s)\,R_k(x-s)\right] ds + \\ + \frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}} \int\limits_0^t \left((t-s)^{-\frac{1}{2}} - t^{-\frac{1}{2}} \right) \frac{d}{dx} \times \\ \times \left[V(x-s)\,R_k(x-s)\right] ds + \frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}} t^{-\frac{1}{2}} V(0), \quad 0 < t < x, \\ \frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}} \left[t^{-\frac{1}{2}} - (t-x)^{-\frac{1}{2}} \right] V(0) + \\ \int\limits_0^t \left((t-s)^{-\frac{1}{2}} - t^{-\frac{1}{2}} \right) \frac{d}{dx} \left[V(x-s)\,R_k(x-s)\right] ds, \quad t > x, \end{cases}$$

which implies

$$\int\limits_0^{+\infty} \left\| D_x A_{2,\nu}^{(0)} \right\| (x,t)\,dt \le$$

$$\le \int\limits_0^x \left[\|U\|(s) + \|Q\|(s) + \|Q'\|(s) + (x-s)\|V'\|(s) \right] \|R_k\|(s) ds + C\sqrt{x}, \quad C > 0.$$

Similarly, the formulas

$$D_x A_{2,\nu}^{(p)}(x,t) = \int\limits_{x-t}^x U(s)\,A_{2,\nu}^{(p-1)}(s, s-x+t)\,ds - \frac{(-1)^\nu}{2i} Q(x) A_{2,\nu}^{(p-1)}(x,t) +$$

$$+ \frac{(-1)^\nu}{2i} \int\limits_0^t D_s \left[Q(s) A_{2,\nu}^{(p-1)}(s,\xi) \right]\Big|_{s=\xi+x-t} d\xi +$$

$$+ \frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}} \int\limits_0^x ds \int\limits_0^t (t-\xi)^{-\frac{1}{2}} D_s \left[V(s)\,A_{2,\nu}^{(p-1)}(s,\xi) \right] d\xi -$$

$$- \frac{\gamma_1^{(\nu)}}{\sqrt{2\pi}} \int\limits_{x-t}^x ds \int\limits_0^{s-x+t} (s-x+t-\xi)^{-\frac{1}{2}} D_s \left[V(s)\,A_{2,\nu}^{(p-1)}(s,\xi) \right] d\xi, \quad 0 < t \le x,$$

$$D_x A_{2,\nu}^{(p)}(x,t) = \int\limits_0^x ds \int\limits_{t-x+s}^t D_s \left[U(s)\,A_{2,\nu}^{(p-1)}(s,\xi) \right] d\xi - \frac{(-1)^\nu}{2i} Q(x) A_{2,\nu}^{(p-1)}(x,t)\,ds +$$

$$+ \frac{(-1)^\nu}{2i} \int\limits_{t-x}^t D_s \left[Q(s) A_{2,\nu}^{(p-1)}(s,\xi) \right]\Big|_{s=\xi+x-t} d\xi -$$

$$-\int_0^x ds \int_0^{t-x+s} \left[(t-x+s-\xi)^{-\frac{1}{2}} - (t-\xi)^{-\frac{1}{2}}\right] D_s \left[q_k(s)\, A_{2,\nu}^{(p-1)}(s,\xi)\right] d\xi, \quad t>x,$$

imply

$$\int_0^{+\infty} \left|D_x A_{2,\nu}^{(p)}(x,t)\right| dt \le \frac{1}{p!}\left(\int_0^x h(x,s)\,ds\right)^{p+1} +$$
$$+C\sqrt{x}\left(\int_0^x h(x,s)\,ds\right)^p, \quad p=0,1,\ldots .$$

Here

$$h(x,s)=\int_0^x \left[\sigma_1(s)+2\sqrt{\frac{2}{\pi}}\,(x-s)^{\frac{1}{2}}\,\sigma_2(s)\right]\|R_k\|(s)ds, \quad 0\le s\le x,$$

$$\sigma_1(s)=\|U\|(s)+\|Q\|(s)+\|Q'\|(s),$$

$$\sigma_2(s)=\|V\|(s)+\|V'\|(s).$$

Therefore we obtain

$$\int_0^{+\infty} \left\|D_x A_{2,\nu}^{(p)}\right\|(x,t)\,dt \le \left\{\int_0^x h(x,s)\,ds + C\sqrt{x}\right\} e^{\int_0^x h(x,s)ds}.$$

Theorem 3 is proved. ◄

Definition 1. *The integral* $\frac{1}{\Gamma(1-\alpha)}\int_a^t (t-u)^{-\alpha} f(x,u)\,du$ *is called a Riemann-Liouville fractional integral of order* α $(0<\alpha<1)$ *with respect to the variable* t *of the function* $f(x,t)$ $(t>a)$ *and it is denoted by* $I_{a,t}^{\alpha} f(x,t)$ *:*

$$I_{a,t}^{\alpha} f(x,t) \overset{def}{=} \frac{1}{\Gamma(\alpha)}\int_a^t (t-u)^{\alpha-1} f(x,u)\,du.$$

Definition 2. *Let* $f(x,.)\in L_1(a\,;+\infty)$ *. The expression* $\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_a^t (t-u)^{-\alpha} f(x,u)\,du$ *is called a Riemann-Liouville fractional derivative of order* $\alpha\,(0<\alpha<1)$ *with respect to the variable* t *of the function* $f(x,t)$ *and it is denoted by* $D_{a,t}^{\alpha} f(x,t)$ *:*

$$D_{a,t}^{\alpha}\, f(x,t) \overset{def}{=} \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_a^t (t-u)^{-\alpha} f(x,u)\,du \;=\frac{\partial}{\partial t} I_{a,t}^{1-\alpha} f(x,u)du.$$

Theorem 4. *If $U(x) \in \mathbf{C}[0,a]$, $V(x), Q(x) \in \mathbf{C}^1[0,a]$ the Riemann-Liouville fractional derivatives $\left(D_{0,t}^{\frac{1}{2}}\right)^{p+1} A_{2,\nu}(x,t)$ $(p=0,1\ \nu=0,1,2,3)$ are integrable on the semi-axis $(0;+\infty)$ for each $x\in[0,\pi]$ and the following equalities hold:*

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}A_{2,\nu}(x,t)+\left(D_{0,t}^{\frac{1}{2}}\right)^2 A_{2,\nu}(x,t)\right)=$$

$$=U(x)A_{2,\nu}(x,t)+\gamma_1^{(\nu)}V(x)\left(D_{0,t}^{\frac{1}{2}}\right)A_{2,\nu}(x,t)+\gamma_2^{(\nu)}Q(x)\left(D_{0,t}^{\frac{1}{2}}\right)^2 A_{2,\nu}(x,t), \quad (38)$$

$$2i(-1)^{\nu+1}\gamma_1^{(\nu)}\frac{d}{dx}\beta_0^{(\nu)}(x)=\gamma_1^{(\nu)}Q(x)\beta_0^{(\nu)}(x)+V(x)R_l(x), \quad (39)$$

$$\frac{d}{dx}\beta_1^{(\nu)}(x)=\gamma_2^{(\nu)}Q(x)\beta_1^{(\nu)}(x)+$$

$$+\left(U(x)+\frac{1}{4}Q^2(x)+\frac{(-1)^\nu}{2i}Q'(x)\right)R_l(x)+\gamma_1^{(\nu)}V(x)\beta_0^{(\nu)}(x). \quad (40)$$

Here

$$\beta_0^{(\nu)}(x)=\left\{I_{0,t}^{\frac{1}{2}}A_{2,\nu}(x,t)\,|_{t=0}\right\},\beta_1^{(\nu)}(x)=\left\{I_{0,t}^{\frac{1}{2}}\left(D_{0,t}^{\frac{1}{2}}\right)A_{2,\nu}(x,t)\,|_{t=0}\right\},$$

$$\gamma_k^{(\nu)}=2^{-\frac{k}{2}}e^{\frac{i\pi k}{4}(2\nu+1)}, l=k+\frac{1}{2}\left((-1)^{k+1}+(-1)^{k+\nu}\right), k=0,1;\nu=0,1,2,3,$$

$$\gamma_2^{(\nu)}=\frac{i}{2}(-1)^\nu.$$

Proof. Integrability of the fractional derivatives $\left(D_{0,t}^{\frac{1}{2}}\right)^{p+1} A_{2,\nu}(x,t)$ on the half line are obtained by application of the fractional integral operators $I_{0,t}^{\frac{1}{2}}$, $I_{0,t}^{\frac{1}{2}}\left(D_{0,t}^{\frac{1}{2}}\right)$ to Eq. (26) and by using Theorem 2 and Theorem 3. Integrability of the fractional derivatives$\left(D_{0,t}^{\frac{1}{2}}\right)^{p+1} A_{2,\nu}(x,t)$ will be also obtained if we prove the formulas (38), (39), and (40). Now let us prove these formulas. If we make the substitution

$$Y_k(x,\lambda)=e^{(-1)^{k+1}i\lambda^2 x}\left[R_k(x)+Z_k(x,\lambda)\right]$$

in Eq. (19), we have

$$Z_k''+2i(-1)^{\nu+1}i\lambda^2 Z_k=$$

$$=(U(x)+\lambda V(x))Z_k+\left(U(x)+\frac{1}{4}Q^2(x)+\frac{(-1)^k}{2i}Q'(x)+\lambda V(x)\right)R_k(x). \quad (41)$$

In the last equation, we put

$$Z_k(x,\lambda)=\int_0^{+\infty}A_{2,\nu}(x,t)\,e^{(-1)^\nu 2i\lambda^2 t}dt, \lambda\in S_\nu. \quad (42)$$

Before doing this, note the following equalities, which are obtained by help of the formula (31) and the properties of the fractional integrals and derivatives:

$$\lambda Z_k(x,\lambda) = \lambda \int\limits_0^{+\infty} A_{2,\nu}\left(x,t\right) e^{(-1)^\nu 2i\lambda^2 t} dt = 2i\lambda^2 \frac{1}{2i\lambda} \int\limits_0^{+\infty} A_{2,\nu}\left(x,t\right) e^{(-1)^\nu 2i\lambda^2 t} dt =$$

$$= 2i\lambda^2 \int\limits_0^{+\infty} A_{2,\nu}\left(x,t\right) dt \left((-1)^{\nu+1} \frac{\gamma_1^{(\nu)}}{\sqrt{\pi}} \int\limits_t^{\infty} (s-t)^{-\frac{1}{2}} e^{(-1)^\nu 2i\lambda^2 s} ds \right) =$$

$$= (-1)^{\nu+1} \frac{\gamma_1^{(\nu)}}{\sqrt{\pi}} 2i\lambda^2 \int\limits_0^{+\infty} e^{(-1)^\nu 2i\lambda^2 s} ds \int\limits_0^{s} (s-t)^{-\frac{1}{2}} A_{2,\nu}\left(x,t\right) dt =$$

$$= (-1)^{\nu+1} \gamma_1^{(\nu)} 2i\lambda^2 \int\limits_0^{+\infty} e^{(-1)^\nu 2i\lambda^2 t} I_{0,t}^{\frac{1}{2}} A_{2,\nu}\left(x,t\right) dt =$$

$$= \gamma_1^{(\nu)} I_{0,t}^{\frac{1}{2}} A_{2,\nu}\left(x,t\right)\Big|_{t=0} + \gamma_1^{(\nu)} \int\limits_0^{+\infty} e^{(-1)^\nu 2i\lambda^2 t} D_{0,t}^{\frac{1}{2}} A_{2,\nu}\left(x,t\right) dt =$$

$$= \gamma_1^{(\nu)} \beta_0^{(\nu)}(x) + \gamma_1^{(\nu)} \int\limits_0^{+\infty} e^{(-1)^\nu 2i\lambda^2 t} D_{0,t}^{\frac{1}{2}} A_{2,\nu}\left(x,t\right) dt,$$

i.e.

$$\lambda Z_k(x,\lambda = \gamma_1^{(\nu)} \beta_0^{(\nu)}(x) + \gamma_1^{(\nu)} \int\limits_0^{+\infty} e^{(-1)^\nu 2i\lambda^2 t} D_{0,t}^{\frac{1}{2}} A_{2,\nu}\left(x,t\right) dt. \tag{43}$$

Analogously, we can write

$$\lambda^2 Z_k(x,\lambda = \lambda\gamma_1^{(\nu)} \beta_0^{(\nu)}(x) + \gamma_2^{(\nu)} \beta_1^{(\nu)}(x) + \gamma_2^{(\nu)} \int\limits_0^{+\infty} e^{(-1)^\nu 2i\lambda^2 t} \left(D_{0,t}^{\frac{1}{2}}\right)^2 A_{2,\nu}\left(x,t\right) dt. \tag{44}$$

As we have mentioned above, from the formulas (43) and (44), we have integrability of the fractional derivatives $D_{0,t}^{\frac{1}{2}} A_{2,\nu}\left(x,t\right)$ and $\left(D_{0,t}^{\frac{1}{2}}\right)^2 A_{2,\nu}\left(x,t\right)$. Now, if we use the formulas (42), (43), and (44) in Eq. (41), we obtain

$$\int\limits_0^{+\infty} \left[D_x(D_x A_{2,\nu}(x,t) + 2i\,(-1)^{\nu+1} \gamma_2^{(\nu)} \left(D_{0,t}^{\frac{1}{2}}\right)^2 A_{2,\nu}\left(x,t\right) \right] e^{(-1)^\nu 2i\lambda^2 t} dt +$$

$$+ 2i\,(-1)^{\nu+1} \left(\lambda\gamma_1^{(\nu)} \beta_0^{(\nu)}(x) + \gamma_2^{(\nu)} \beta_1^{(\nu)}(x) \right)' =$$

$$= \left(U(x) + \frac{1}{4}Q^2(x) + \frac{(-1)^k}{2i}Q'(x) + \lambda V(x) \right) R_k(x) +$$

$$+\gamma_1^{(\nu)}\beta_0^{(\nu)}(x)V(x) + \gamma_2^{(\nu)}\beta_1^{(\nu)}(x)Q(x) + \lambda\gamma_1^{(\nu)}\beta_0^{(\nu)}(x)Q(x) +$$

$$+ \int_0^{+\infty} \left[U(x)A_{2,\nu}(x,t) + \gamma_1^{(\nu)}V(x)D_{0,t}^{\frac{1}{2}}A_{2,\nu}(x,t) + \gamma_2^{(\nu)}Q(x)\left(D_{0,t}^{\frac{1}{2}}\right)^2 A_{2,\nu}(x,t) \right] e^{(-1)^\nu 2i\lambda^2 t}dt.$$

Since

$$2i\,(-1)^{\nu+1}\gamma_2^{(\nu)}\frac{d}{dx}\beta_1^{(\nu)}(x) =$$

$$= \gamma_1^{(\nu)}\beta_0^{(\nu)}(x)V(x) + \gamma_2^{(\nu)}\beta_1^{(\nu)}(x)Q(x) + \left(U(x) + \frac{1}{4}Q^2(x) + \frac{(-1)^k}{2i}Q'(x \right) R_k(x) \quad (45)$$

and

$$2i\,(-1)^{\nu+1}\lambda\gamma_1^{(\nu)}\frac{d}{dx}\beta_0^{(\nu)}(x) = \gamma_1^{(\nu)}\beta_0^{(\nu)}(x)Q(x) + V(x)R_k(x) \quad (46)$$

we obtain

$$\int_0^{+\infty} \left[D_x(D_xA_{2,\nu}(x,t) + 2i\,(-1)^{\nu+1}\gamma_2^{(\nu)}\left(D_{0,t}^{\frac{1}{2}}\right)^2 A_{2,\nu}(x,t) \right] e^{(-1)^\nu 2i\lambda^2 t}dt =$$

$$= \int_0^{+\infty} \left[U(x)A_{2,\nu}(x,t) + \gamma_1^{(\nu)}V(x)D_{0,t}^{\frac{1}{2}}A_{2,\nu}(x,t) + \gamma_2^{(\nu)}Q(x)\left(D_{0,t}^{\frac{1}{2}}\right)^2 A_{2,\nu}(x,t) \right] e^{(-1)^\nu 2i\lambda^2 t}. \quad (47)$$

Now the formulas (38)-(40) are obtained from (45)-(47). Theorem is proved. ◄

Similarly, we can prove the following theorem:

Theorem 5. *If* $U(x) \in \mathbf{C}[0,a]$, $V(x), Q(x) \in \mathbf{C}^1[0,a]$ *the Riemann-Liouville fractional derivatives* $\left(D_{-x,t}^{\frac{1}{2}}\right)^{p+1} A_{1,\nu}(x,t)$ $(p = 0,1\ \nu = 0,1,2,3)$ *are integrable on the semiaxis* $(-x;+\infty)$ *for each* $x \in [0,\pi]$ *and the following equalities hold:*

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}A_{1,\nu}(x,t) - \left(D_{-x,t}^{\frac{1}{2}}\right)^2 A_{1,\nu}(x,t) \right) =$$

$$= U(x)A_{1,\nu}(x,t) + \gamma_1^{(\nu)}V(x)\left(D_{-x,t}^{\frac{1}{2}}\right)A_{1,\nu}(x,t) +$$

$$+\gamma_2^{(\nu)}Q(x)\left(D_{-x,t}^{\frac{1}{2}}\right)^2 A_{1,\nu}(x,t),$$

$$2i\,(-1)^\nu\gamma_1^{(\nu)}\frac{d}{dx}\alpha_0^{(\nu)}(x) = \gamma_1^{(\nu)}Q(x)\,\alpha_0^{(\nu)}(x) + V(x)\,R_l(x),$$

$$-\frac{d}{dx}\alpha_1^{(\nu)}(x) = \gamma_2^{(\nu)}Q(x)\alpha_1^{(\nu)}(x) +$$

$$+\left(U(x)+\frac{1}{4}Q^2(x)+\frac{(-1)^{\nu+1}}{2i}Q'(x)\right)R_l\left(x\right)+\gamma_1^{(\nu)}V\left(x\right)\alpha_0^{(\nu)}(x).$$

Here

$$\alpha_0^{(\nu)}(x)=\left\{I_{-x,t}^{\frac{1}{2}}A_{1,\nu}\left(x,t\right)|_{t=0^+}-I_{-x,t}^{\frac{1}{2}}A_{1,\nu}\left(x,t\right)|_{t=0^-}\right\},$$

$$\alpha_1^{(\nu)}(x)=\left\{I_{-x,t}^{\frac{1}{2}}\left(D_{-x,t}^{\frac{1}{2}}\right)A_{1,\nu}\left(x,t\right)|_{t=0^+}-I_{-x,t}^{\frac{1}{2}}\left(D_{-x,t}^{\frac{1}{2}}\right)A_{1,\nu}\left(x,t\right)|_{t=0^-}\right\}.$$

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